Min-k-planar Drawings of Graphs

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University of Ottawa, Canada
Carleton University, Canada

GD 2023
September 20th

Motivation and definition

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k-planar:
Every edge has at most k crossings

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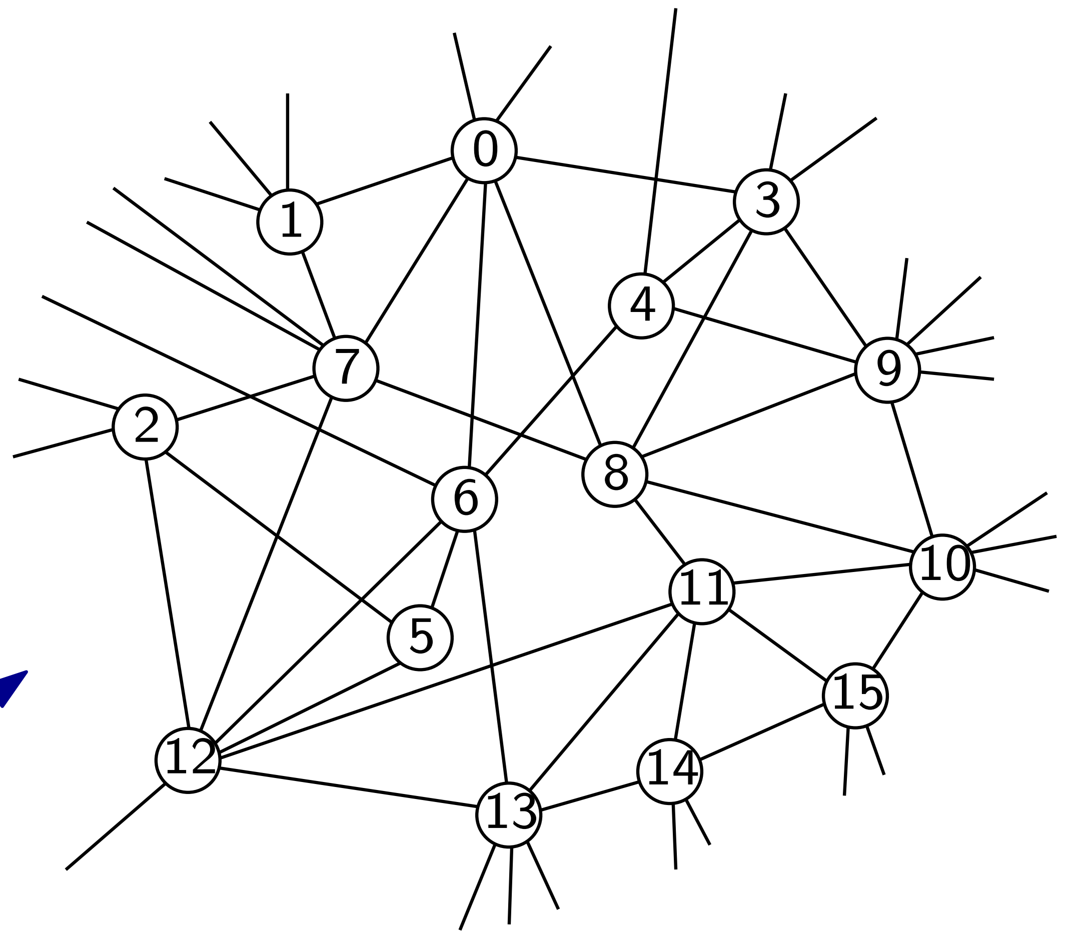

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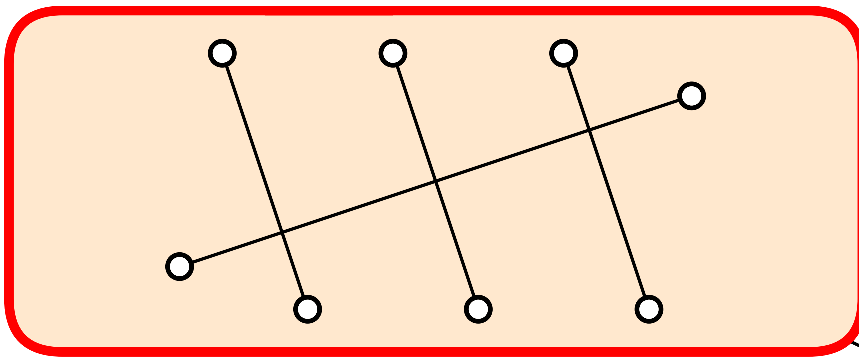

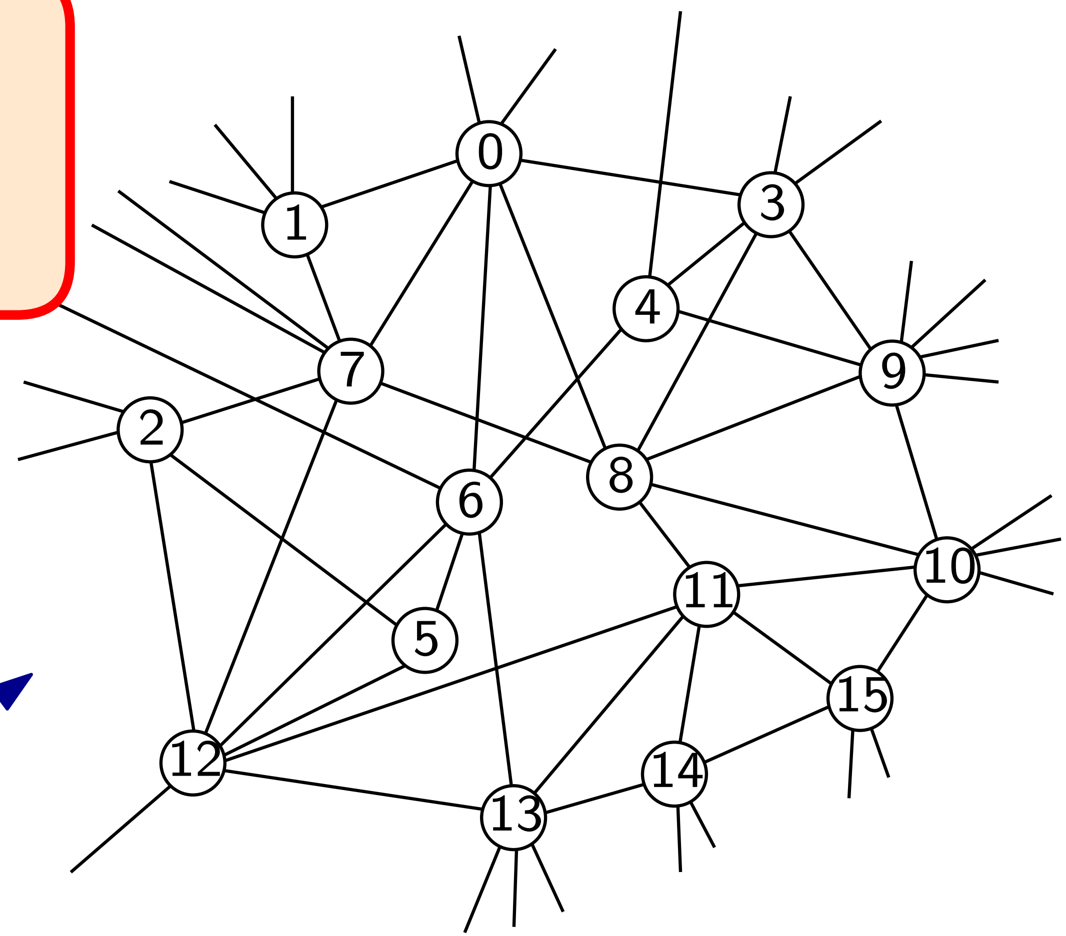

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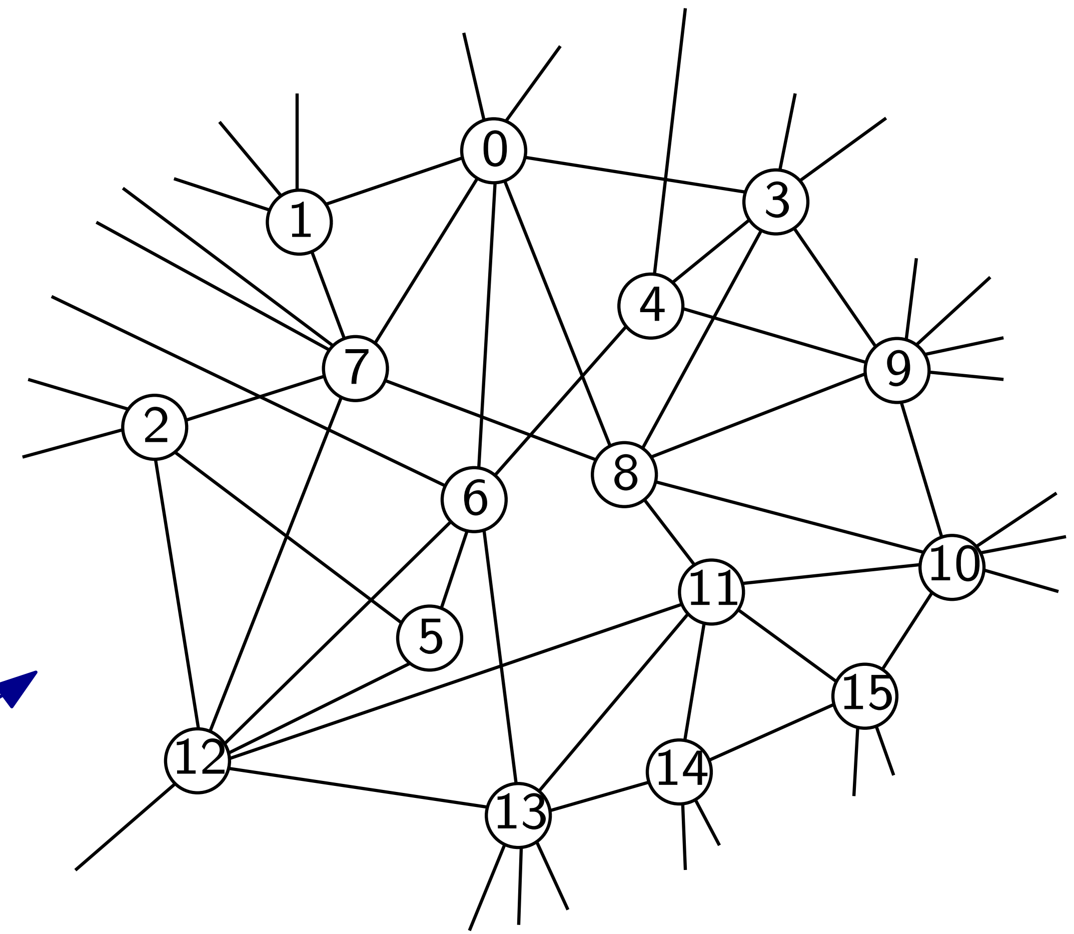

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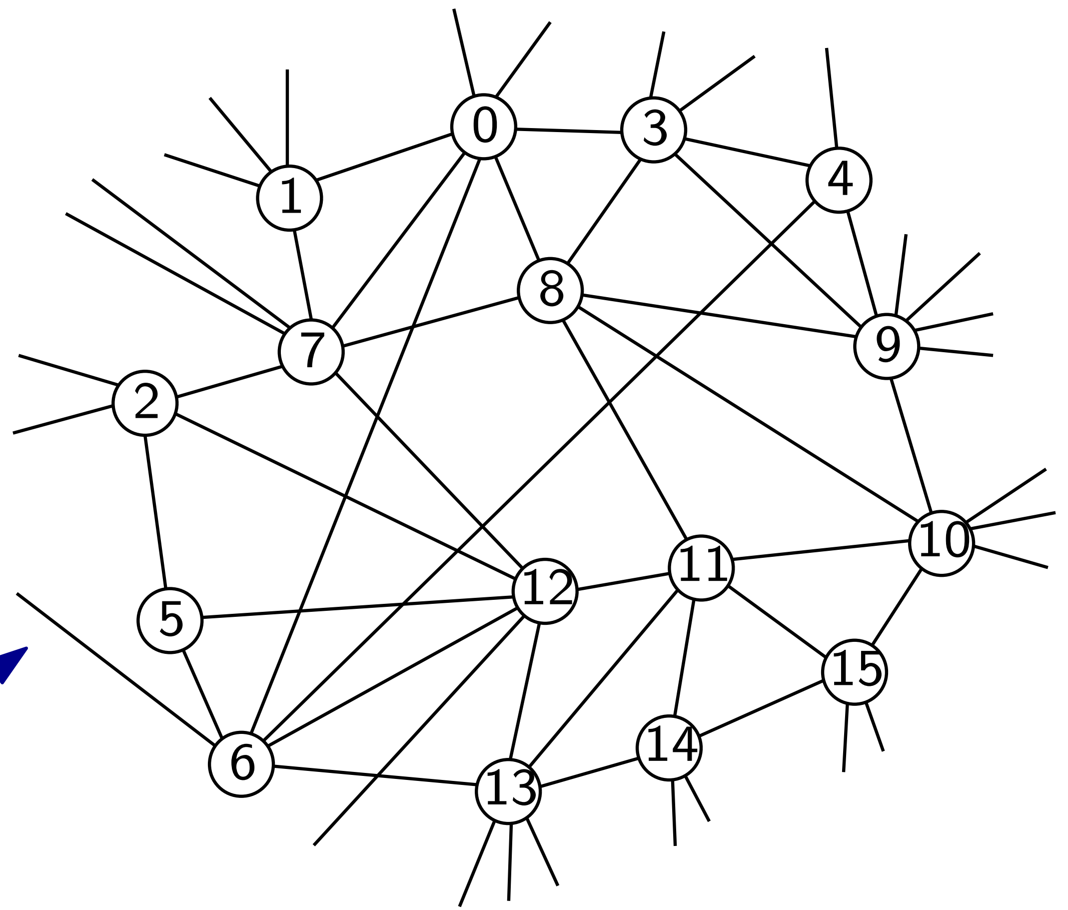

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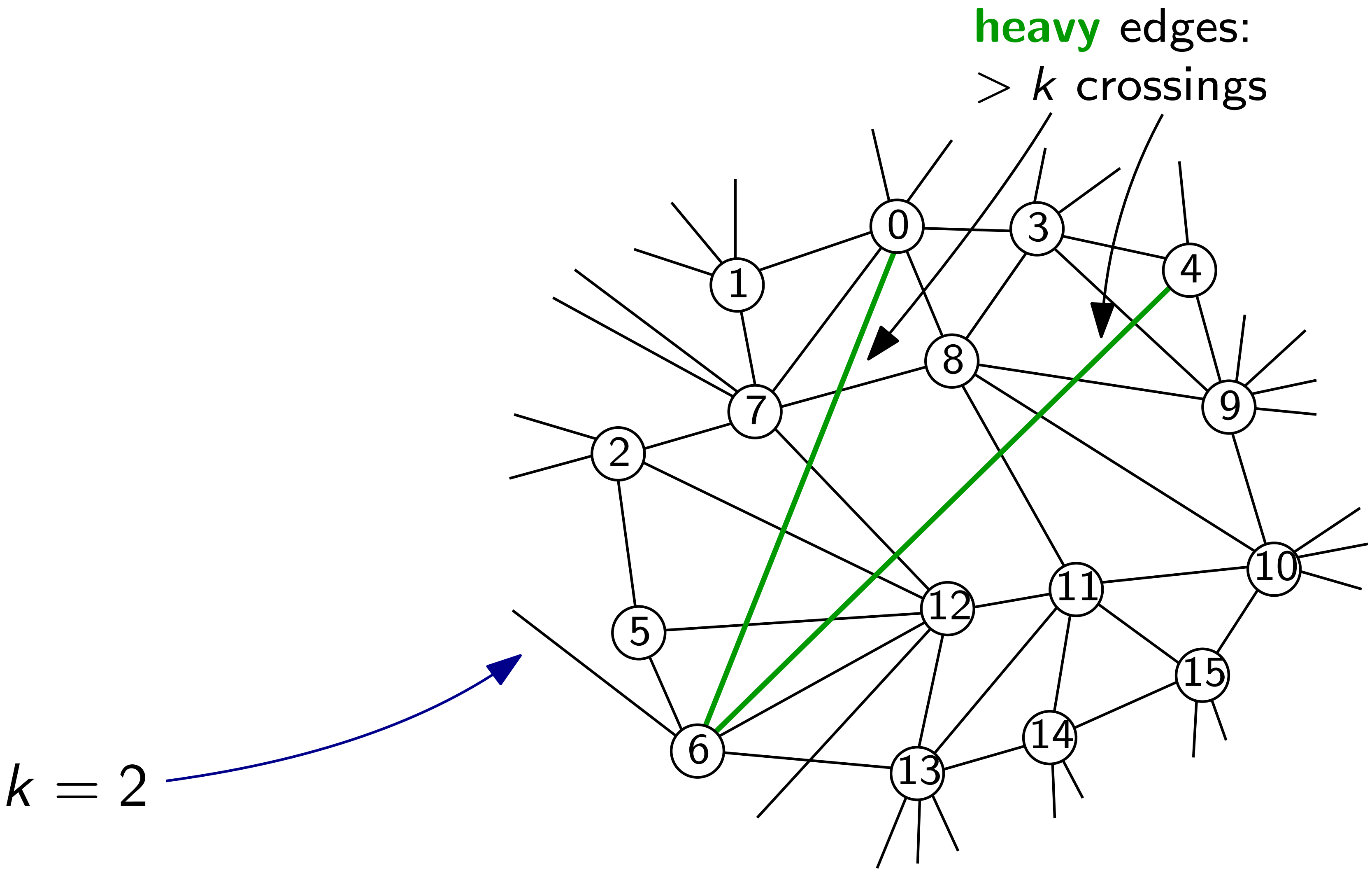

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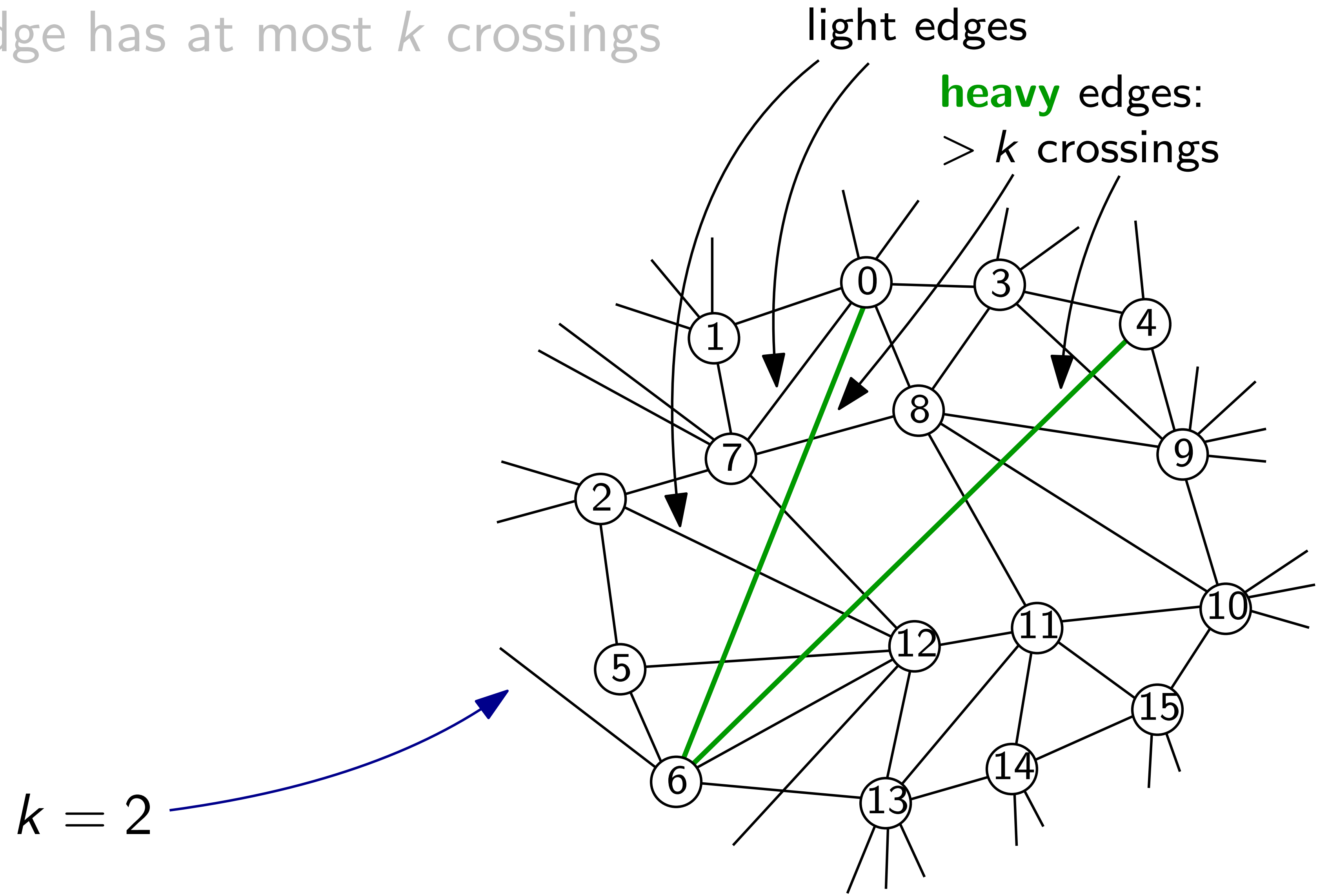

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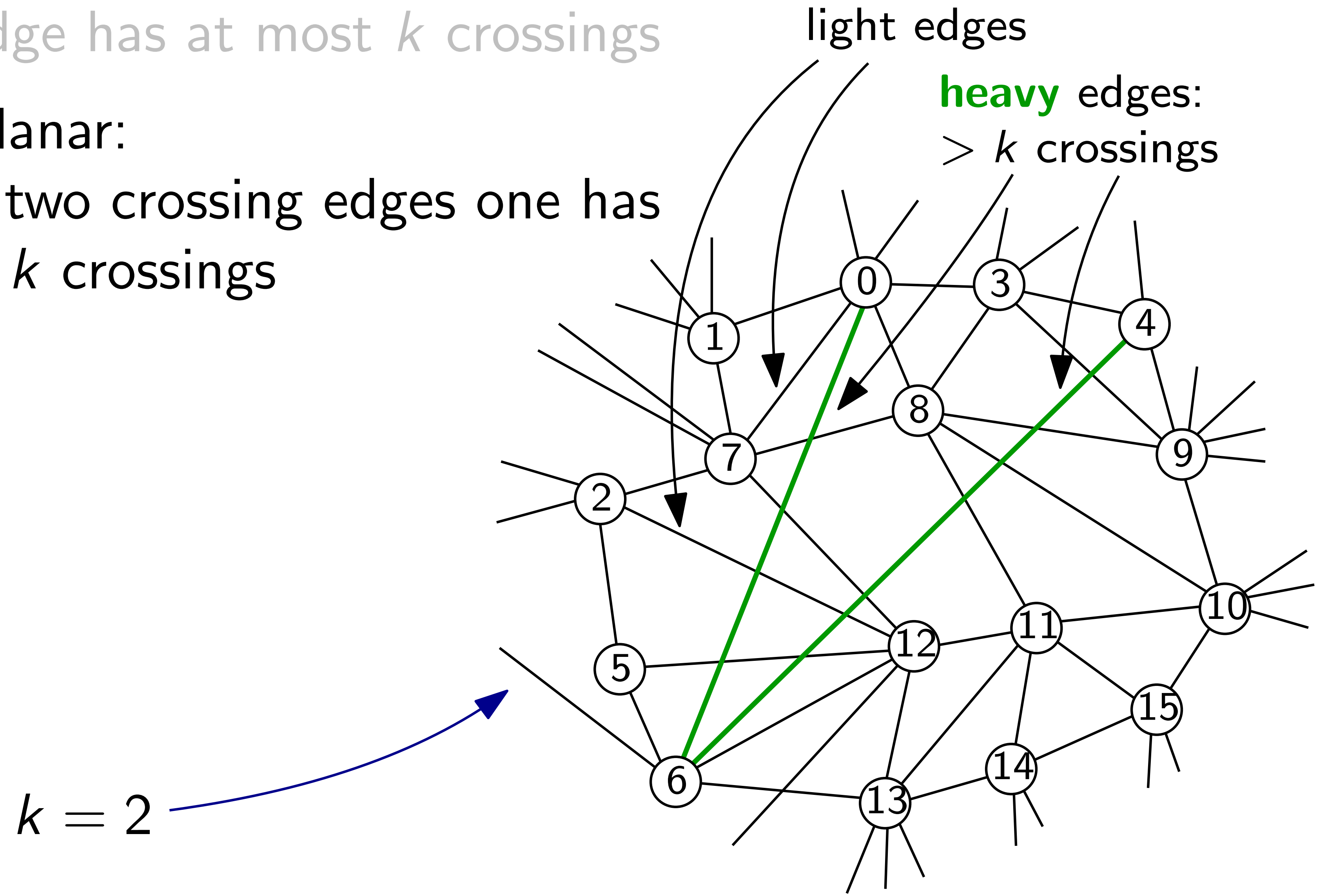

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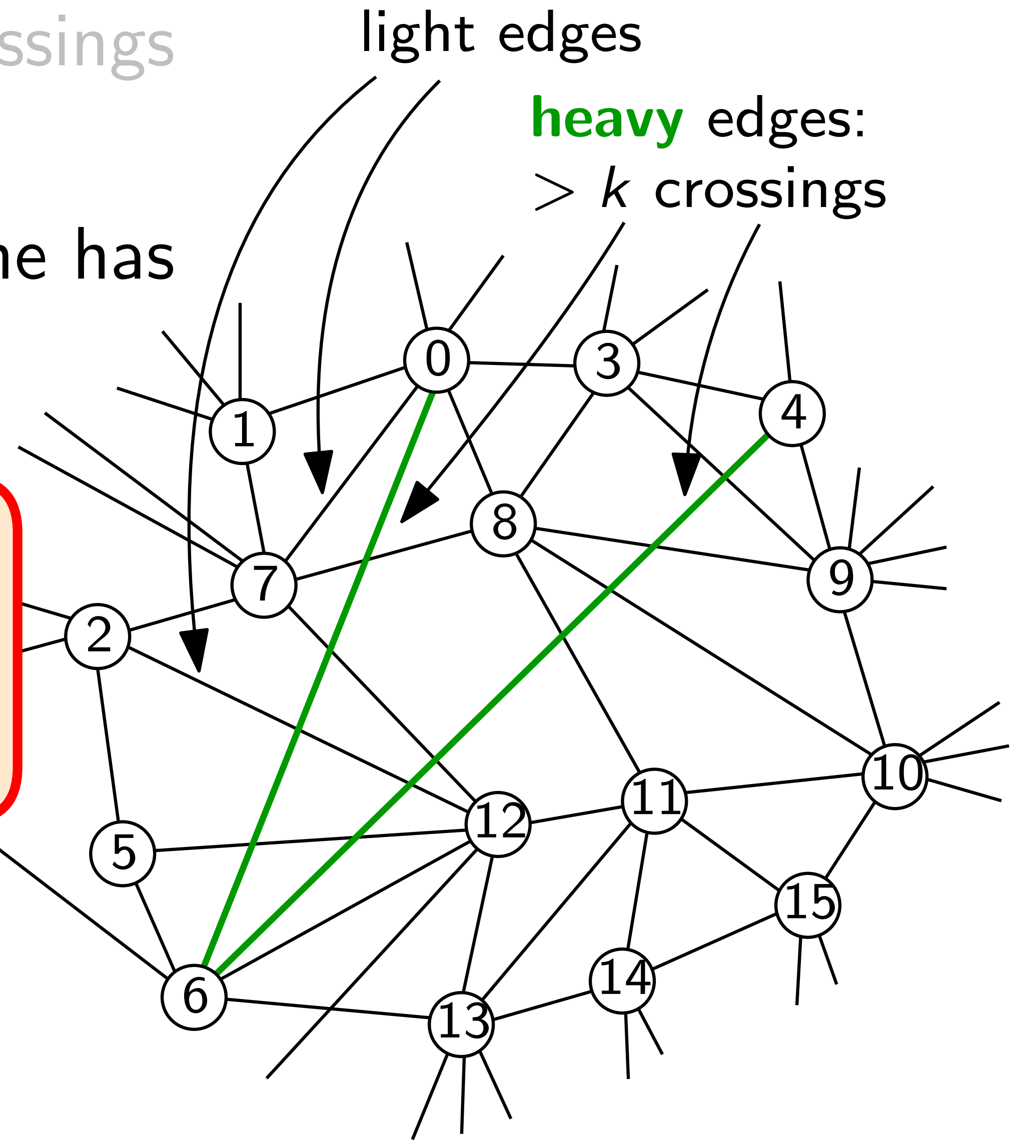

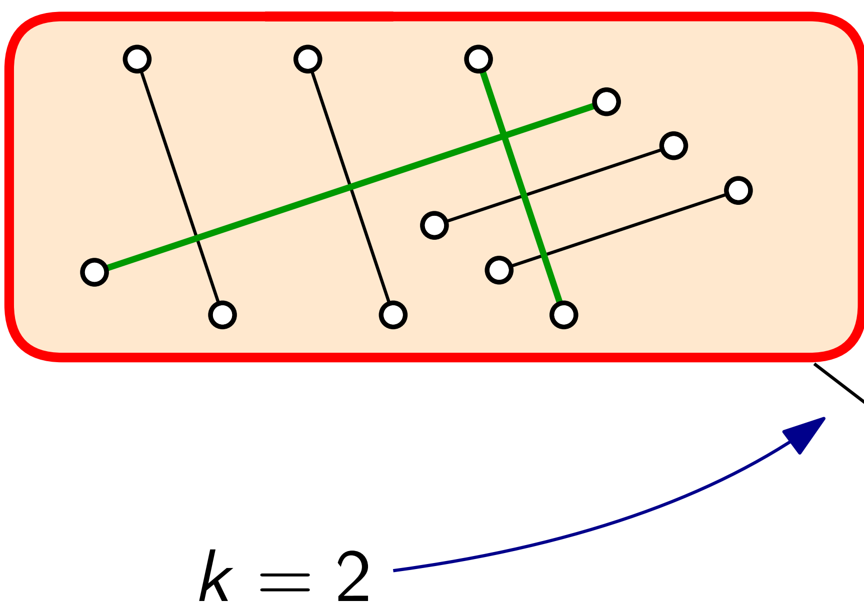

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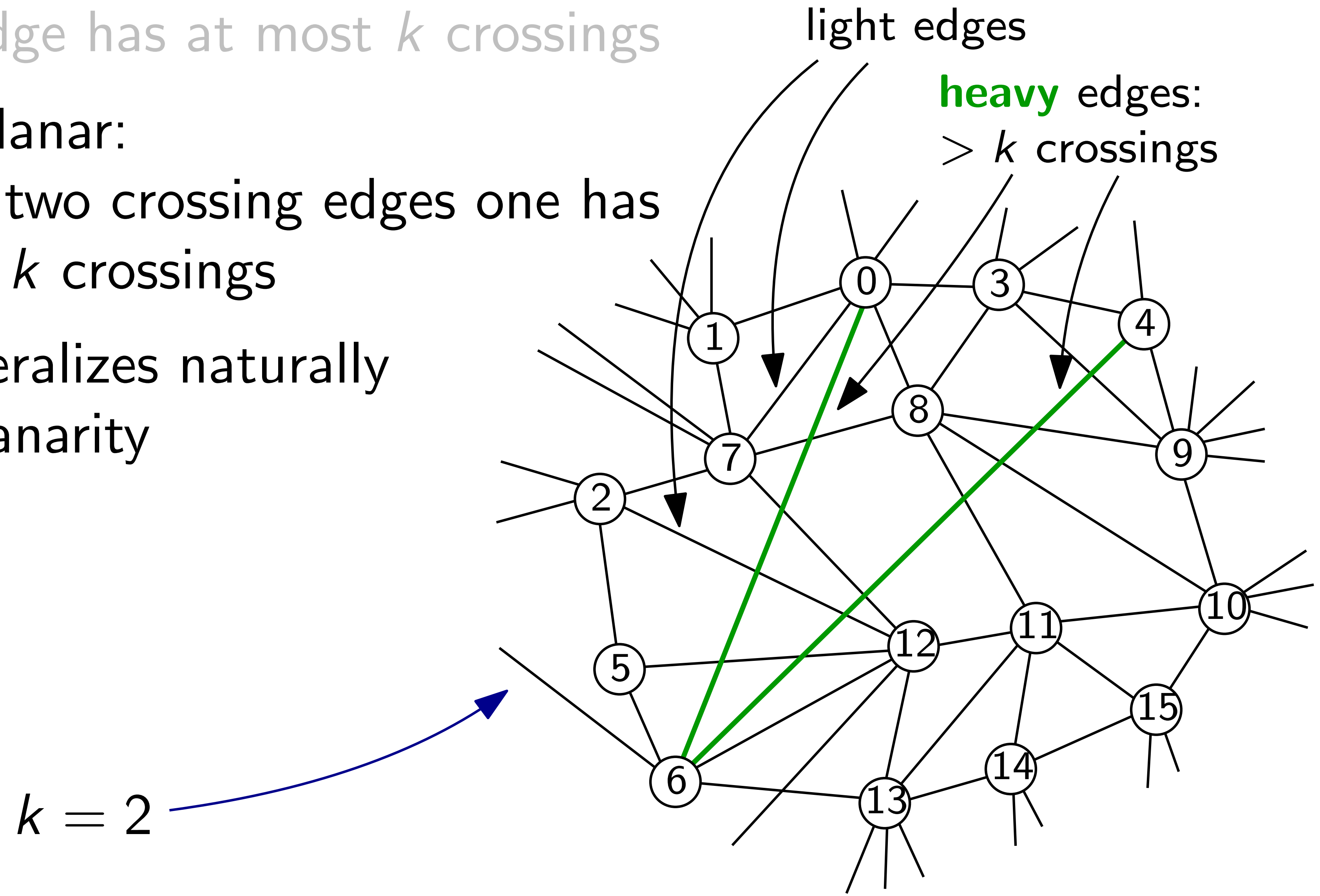
$\Rightarrow$ Generalizes naturally *k*-planarity



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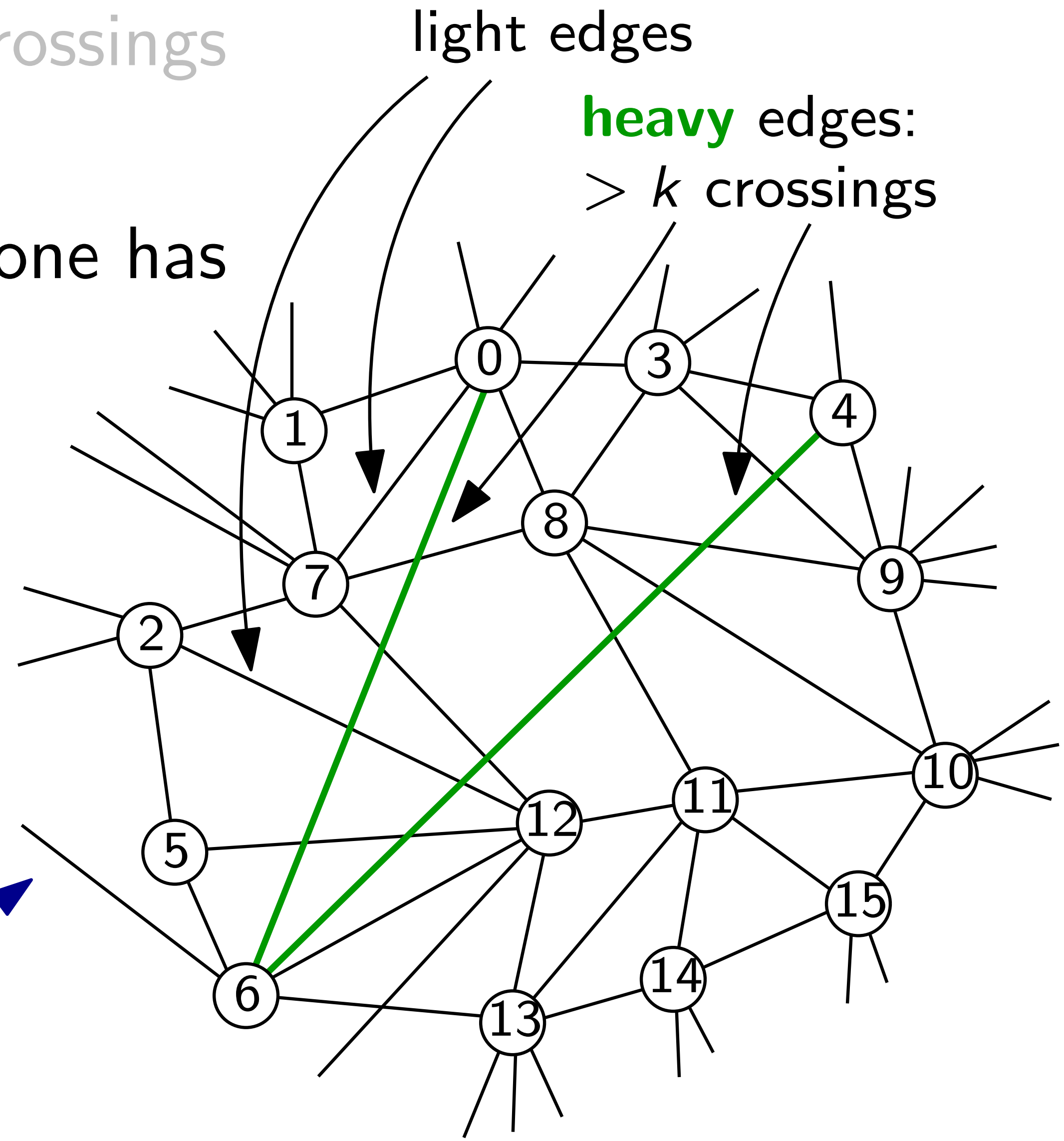

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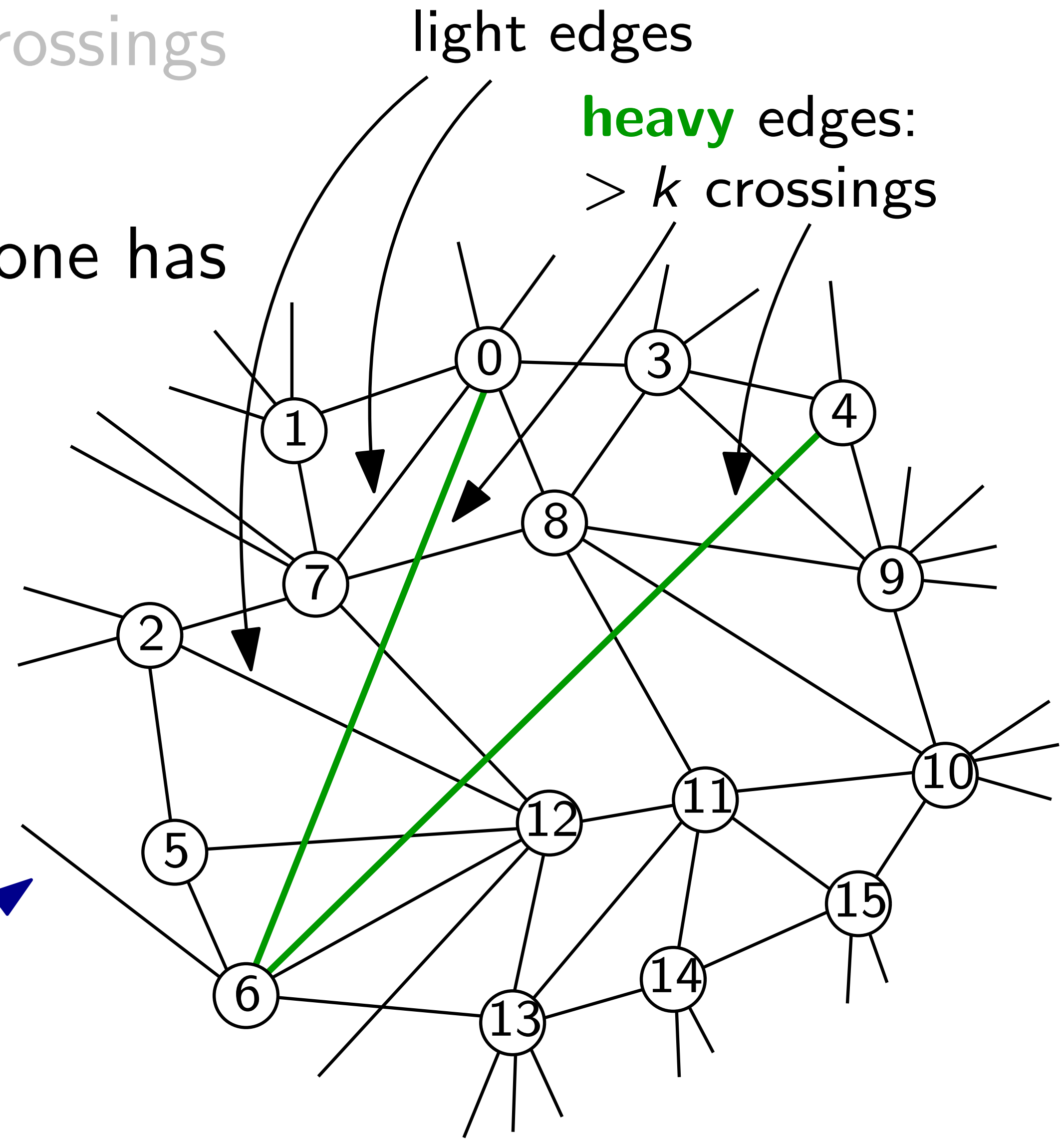

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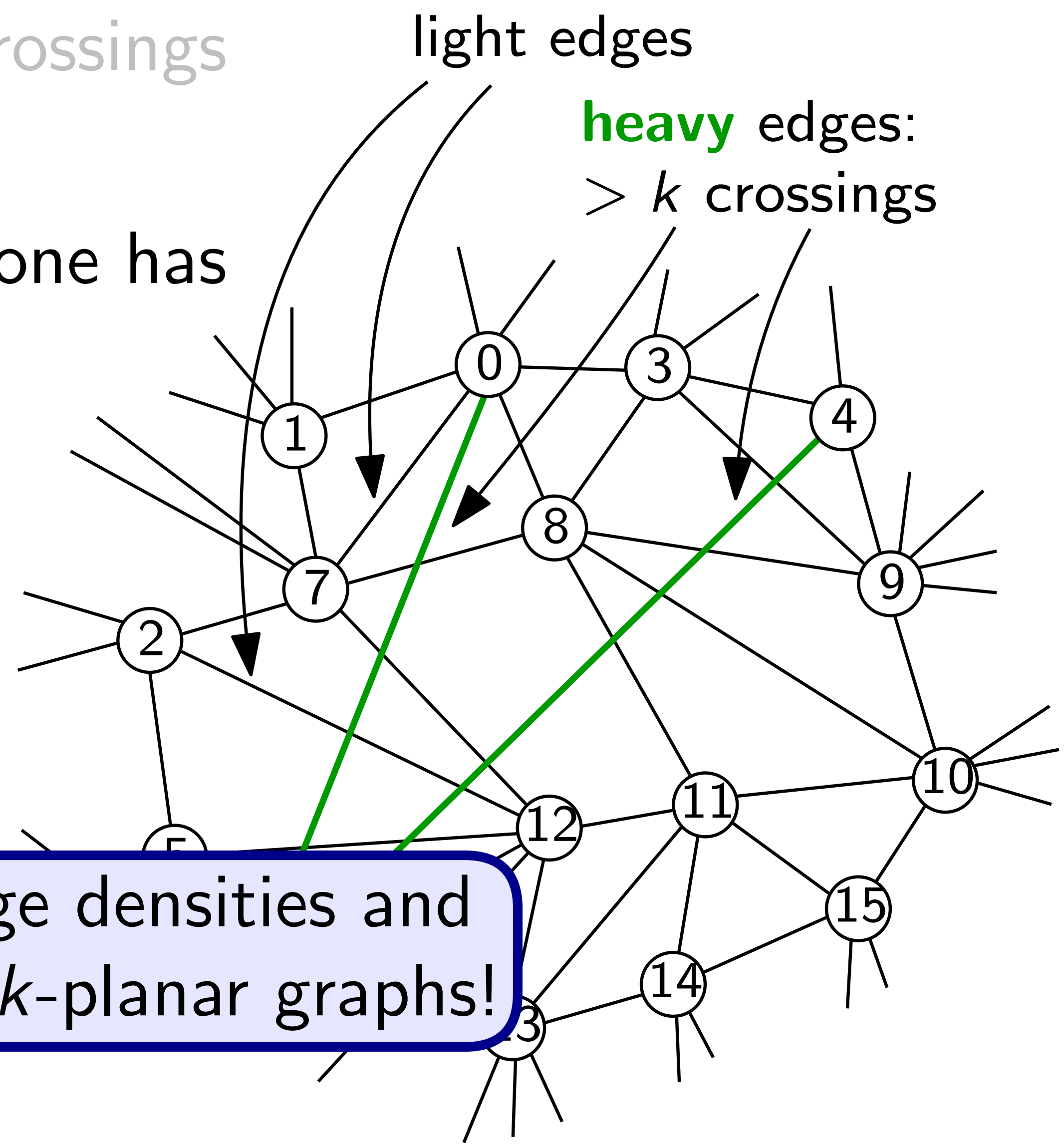

We focus on edge densities and relationships to *k*-planar graphs!

Recap: Simple graphs and drawings

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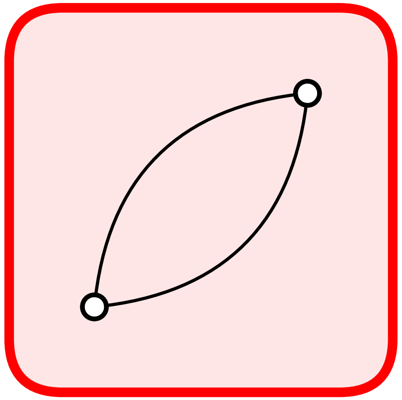

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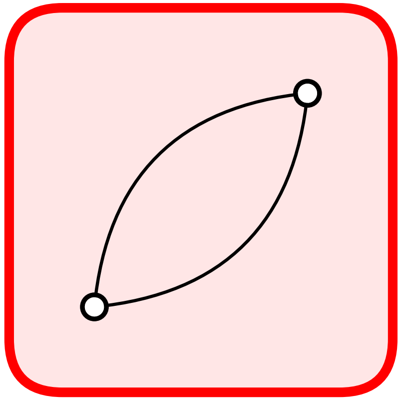

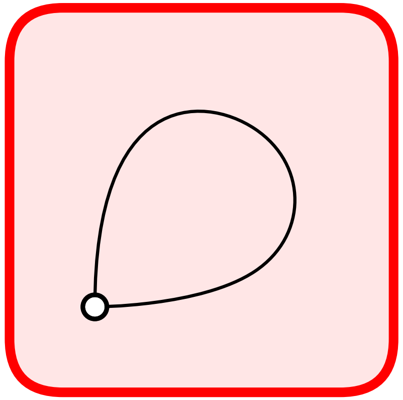

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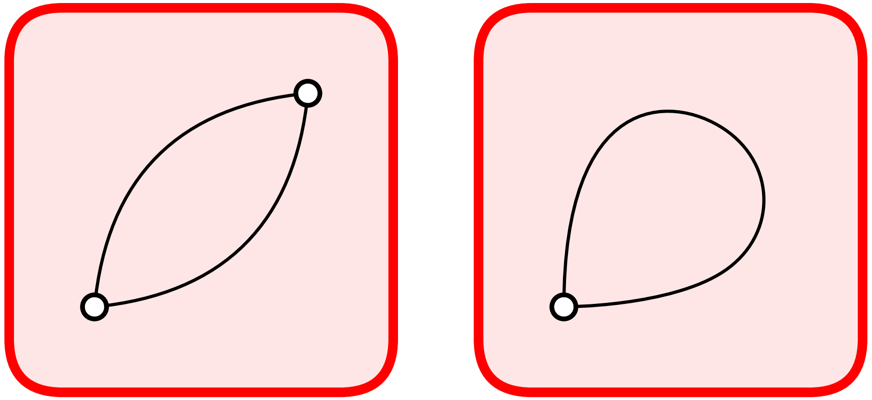

Simple drawings:

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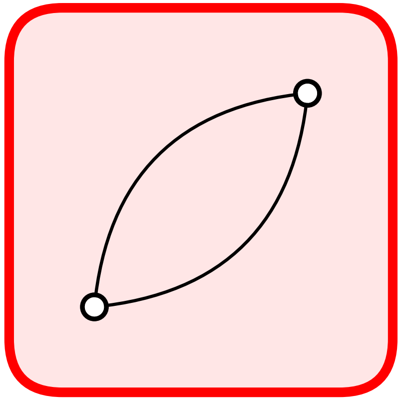

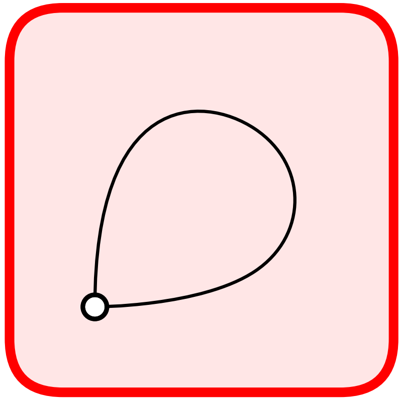

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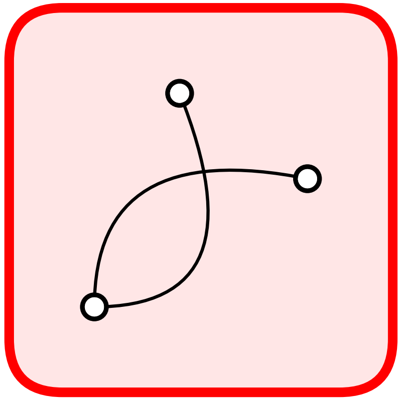

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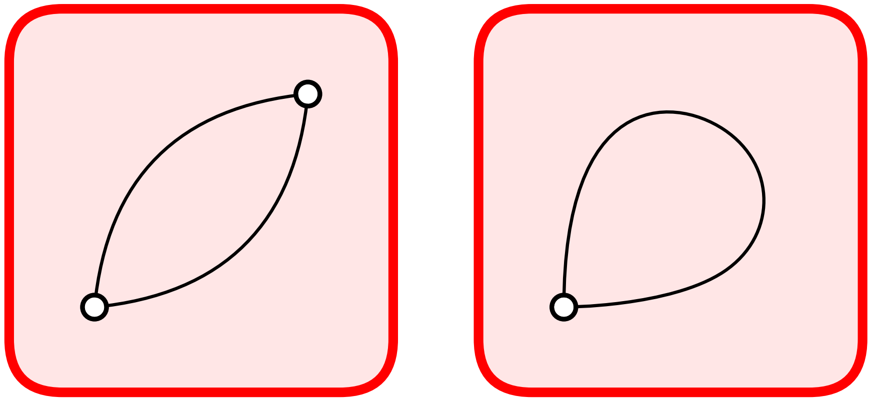

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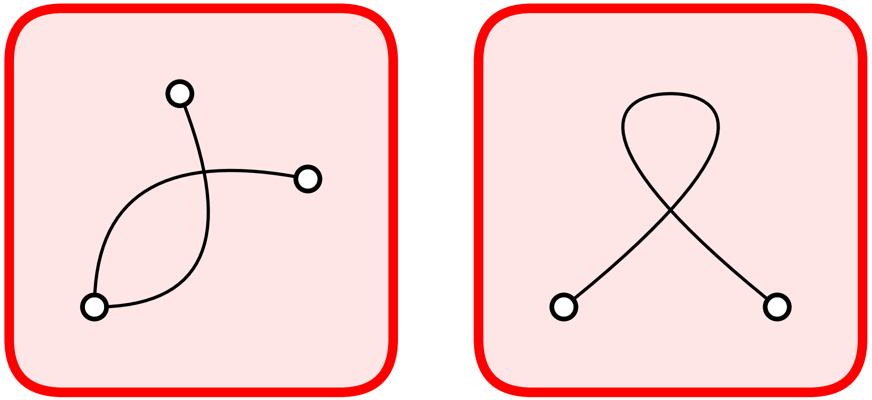

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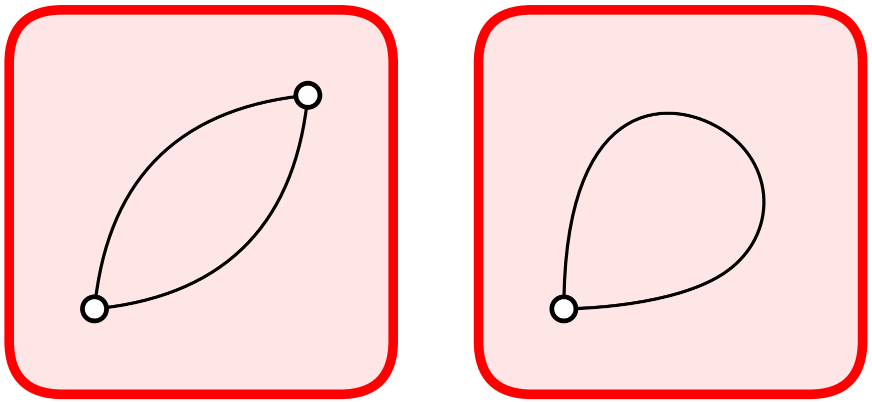

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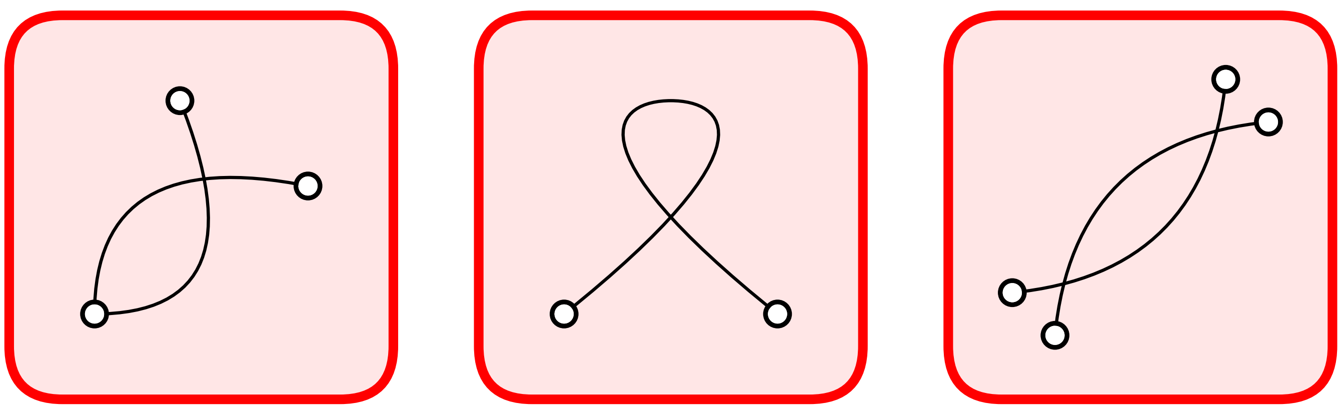

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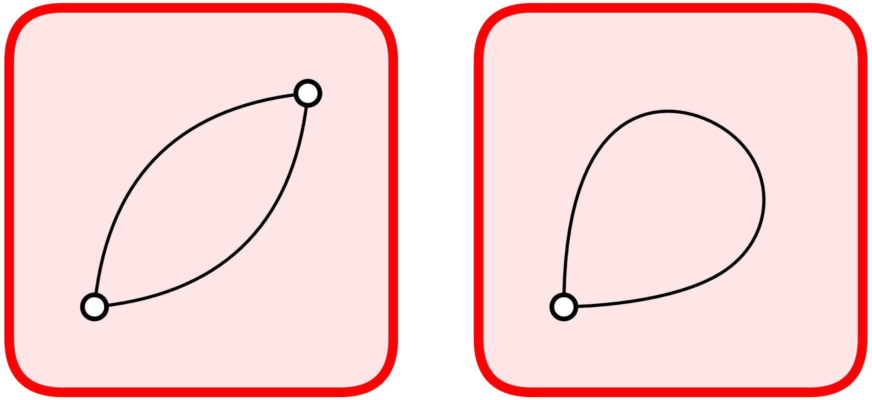

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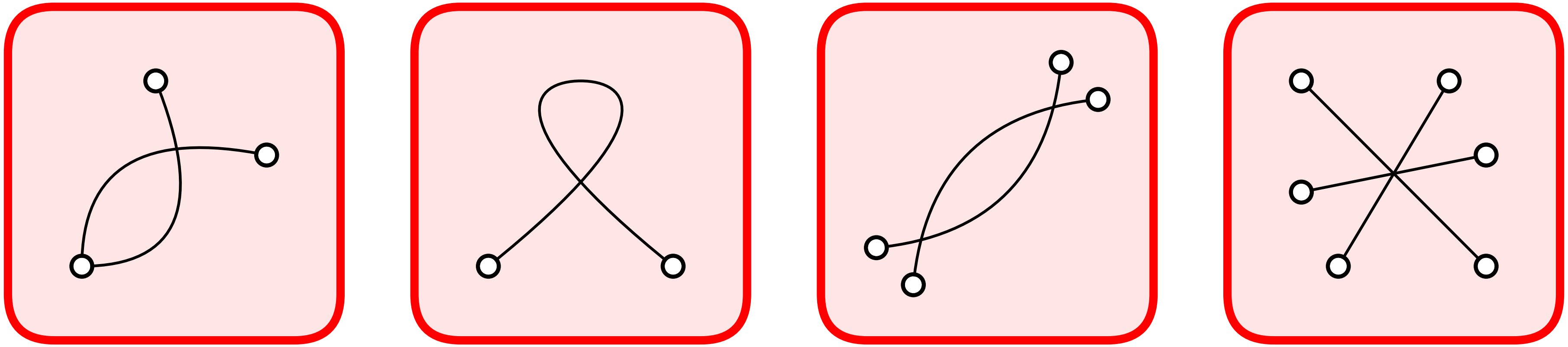

Edge density: General bound

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$$E = E_{light} \cup E_{heavy}$$

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$\Rightarrow \leq 3.81\sqrt{k} \cdot n$ edges [Ackerman '19]

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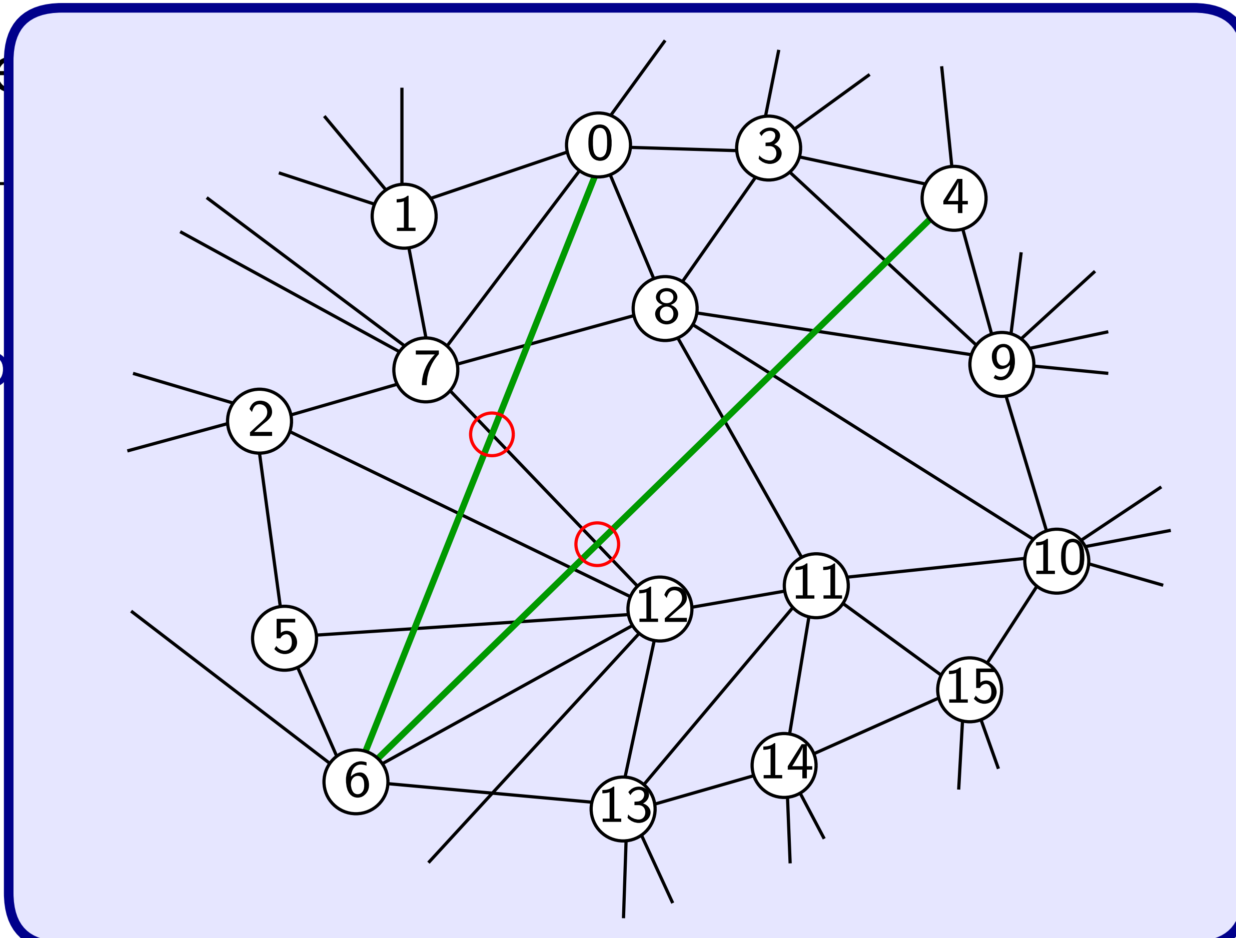

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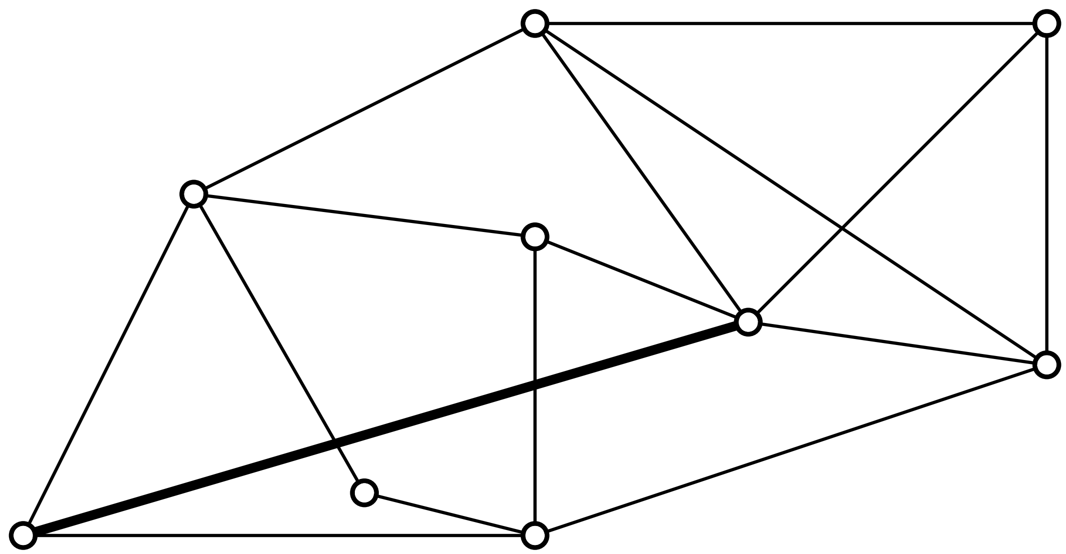

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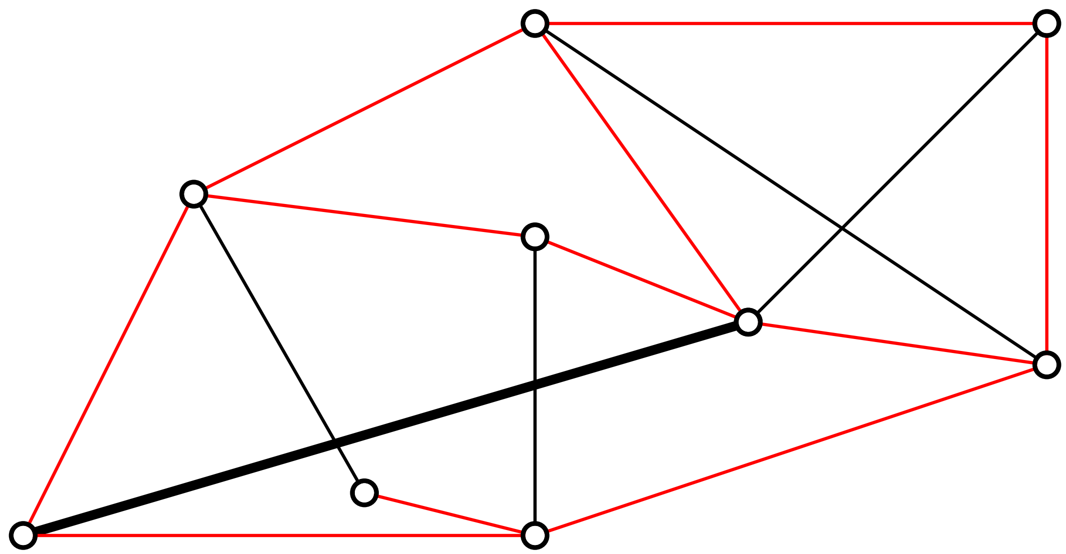

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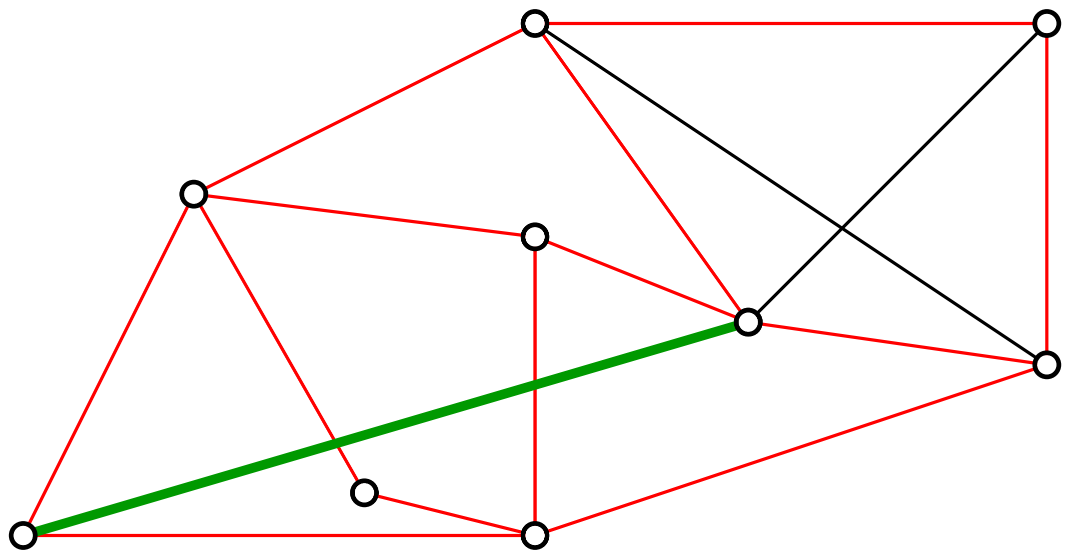

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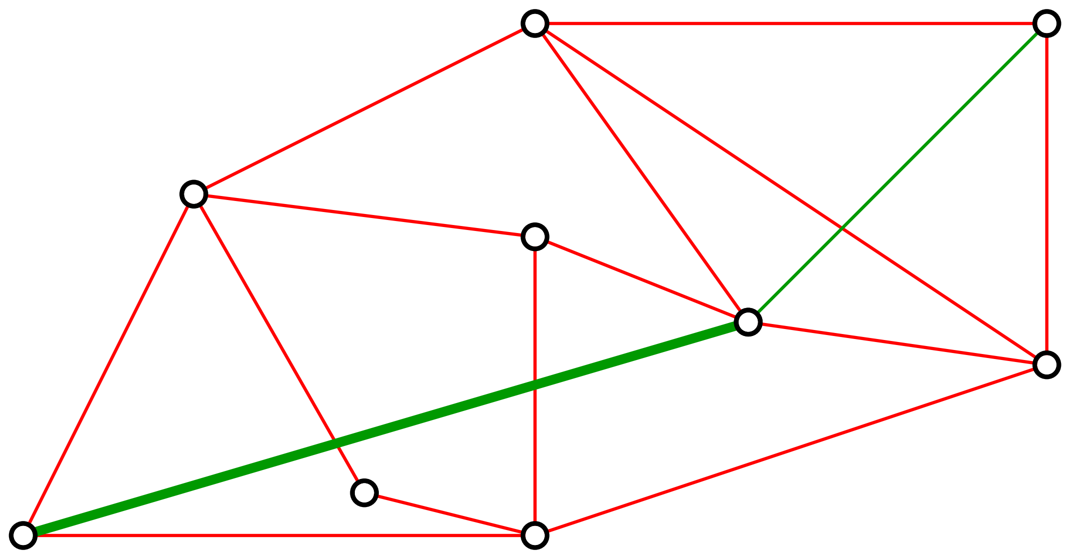

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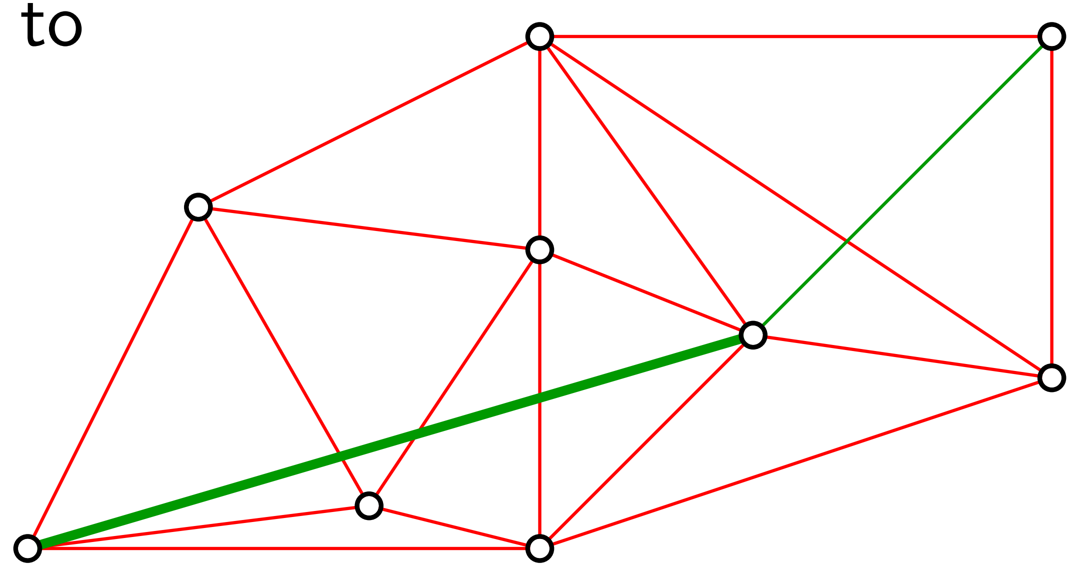

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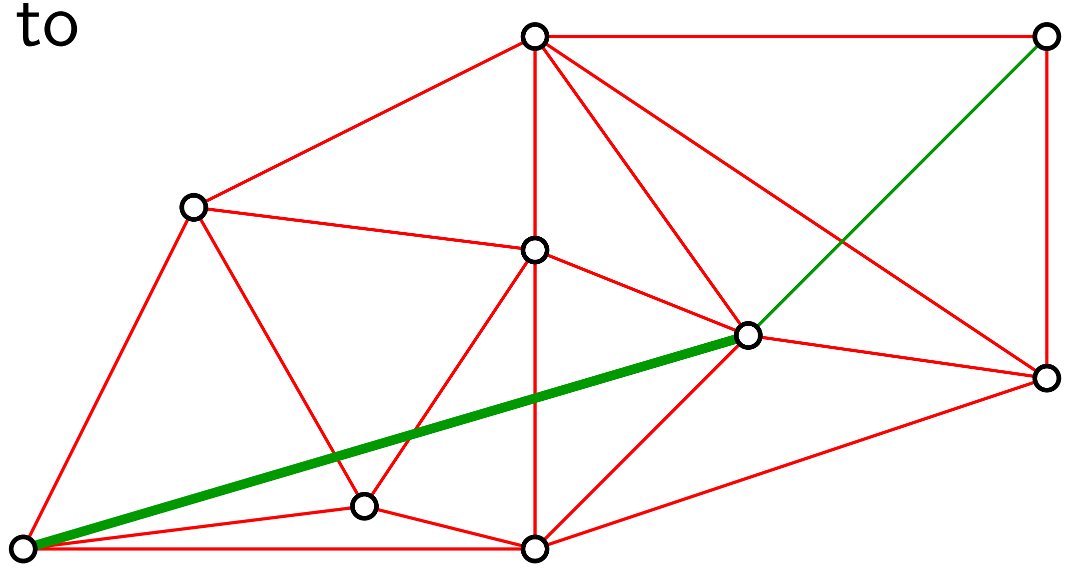

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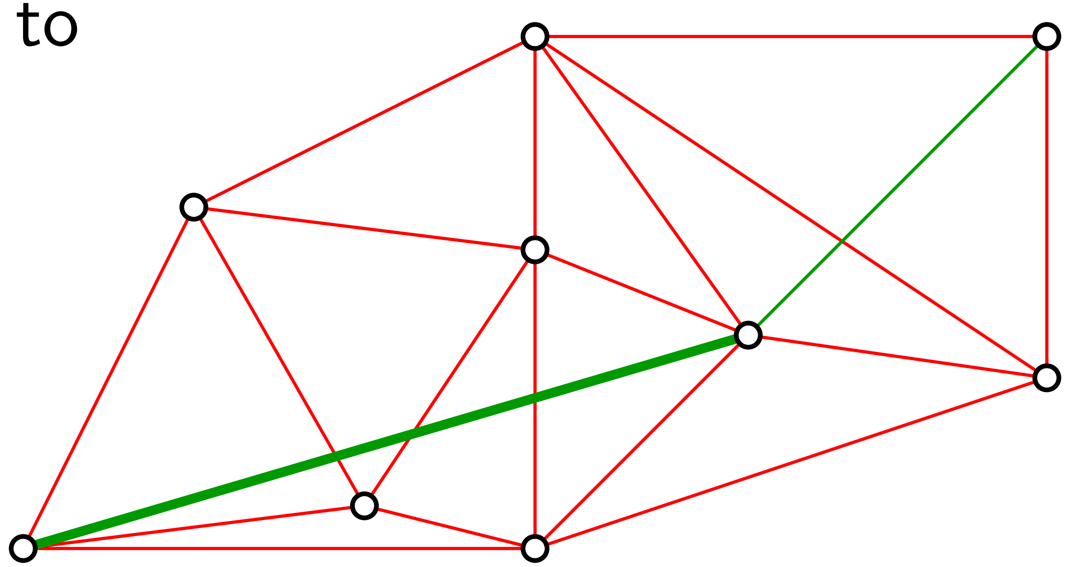
$\Rightarrow \leq 3(n-2)$ edges

$\Rightarrow \leq 2(n-2)$ faces



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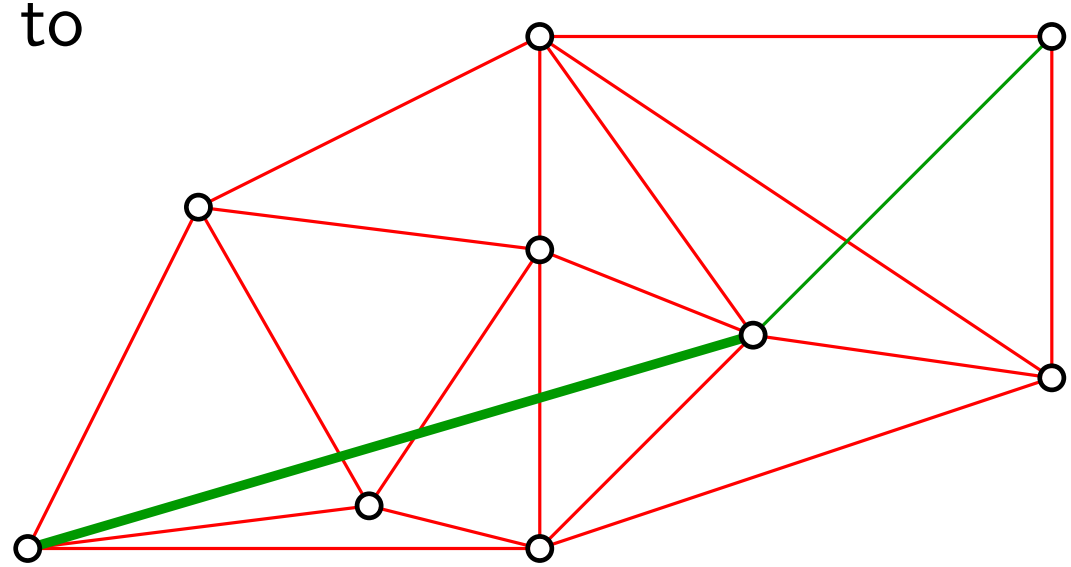

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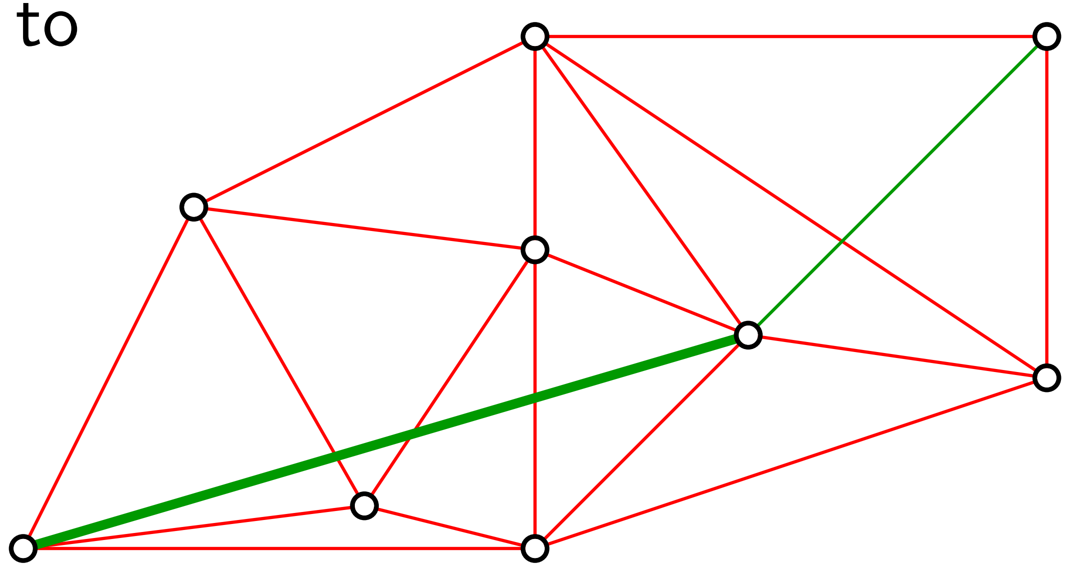

Tight: Optimal 1-planar graphs

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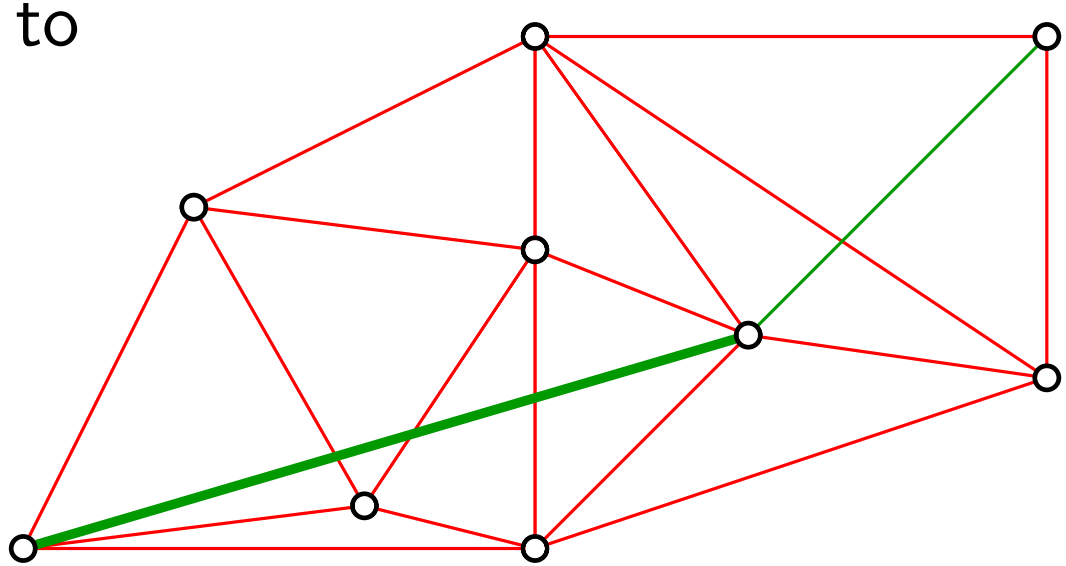


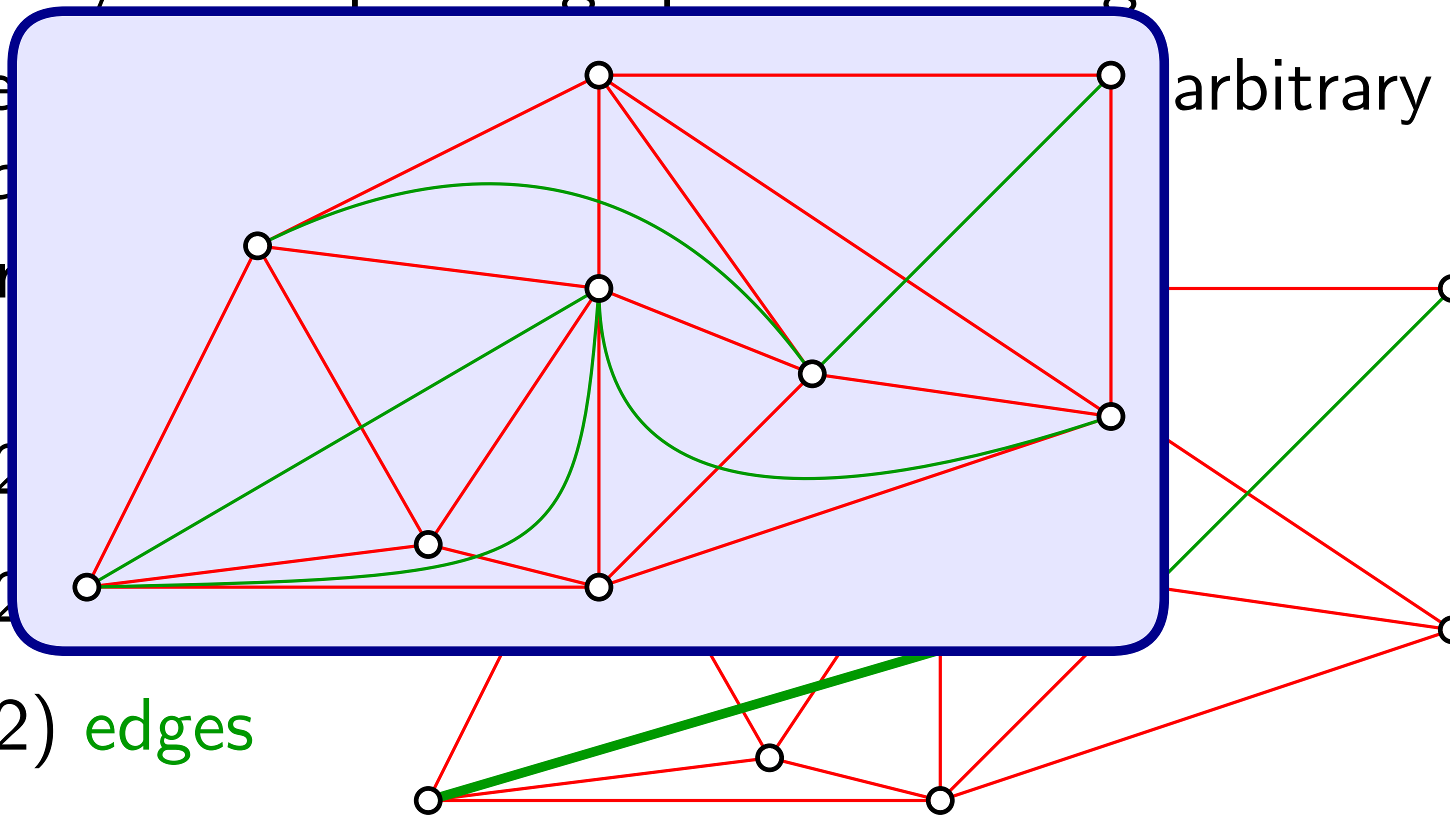
Tight: Optimal 1-planar graphs ($=$ optimal min-1-planar)

Edge density: $k = 1$

Theorem: min-1-planar graphs have $\leq 4(n-2)$ edges, which is a tight bound

Proof:

- Take arbitrary min-1-planar graph and drawing
- Color edge arbitrary if $cr(e_1) = c$
- Augment r planar tria



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- Tight: Optimal 2-planar graphs

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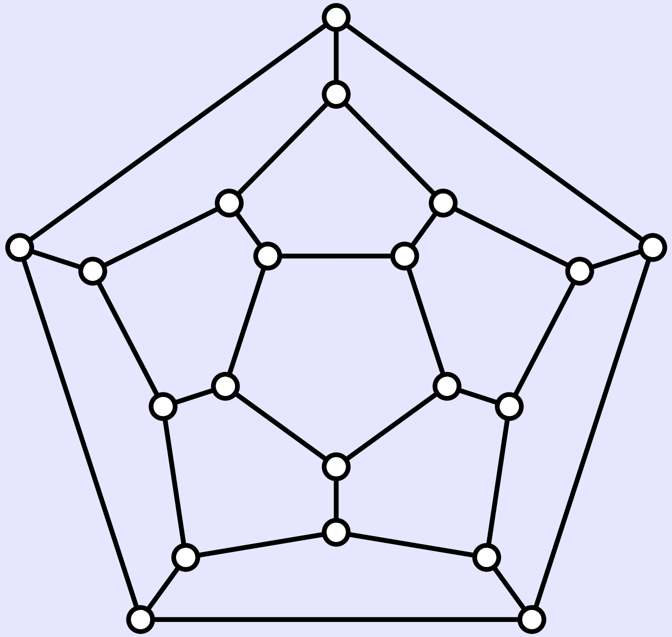

planar structure:

$n = 20$
$e = 30$
$f = 12$

Edge densities: $k = 2$ and $k = 3$

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- D
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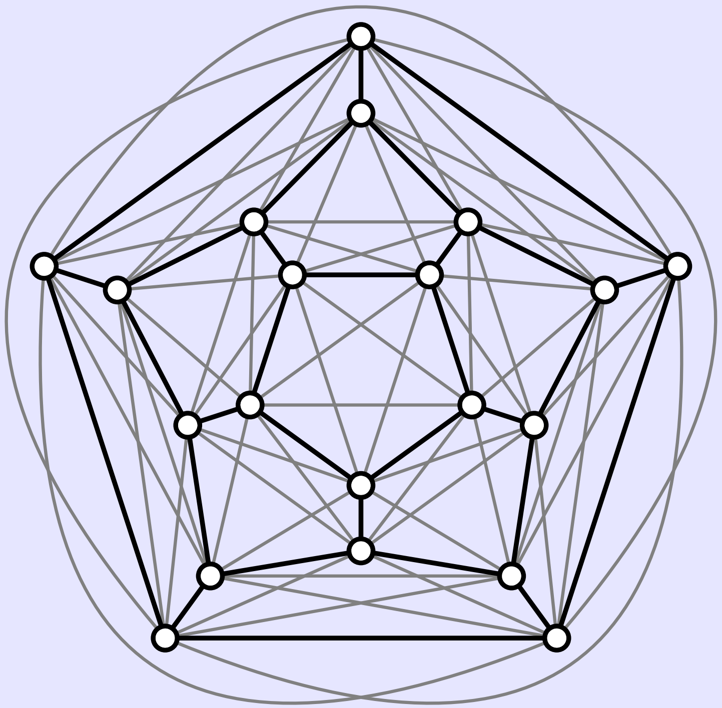

planar structure:

$n = 20$

$e = 30$

$f = 12$

filled by:

5 edges / face

Edge densities: $k = 2$ and $k = 3$

Theorem: min-2-planar graphs have $\leq 5(n-2)$ edges, which is a tight bound

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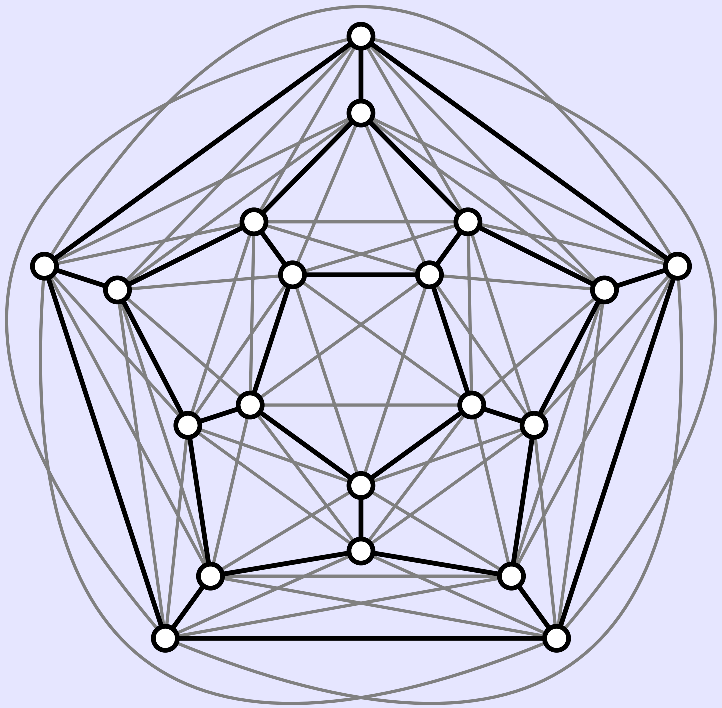

planar structure:

$n = 20$

$e = 30$

$f = 12$

filled by:

5 edges / face

$\Rightarrow$ 90 edges

Edge densities: $k = 2$ and $k = 3$

Theorem: min-2-planar graphs have $\leq 5(n-2)$ edges, which is a tight bound

Proof:

- Discharging technique
- Tight: Optimal 2-planar graphs

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Theorem: min-2-planar graphs have $\leq 5(n-2)$ edges, which is a tight bound

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Theorem: min-3-planar graphs have $\leq 6(n-2)$ edges

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Proof:

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- Not tight: lower bound $5.5n - O(1)$ (optimal 3-planar)

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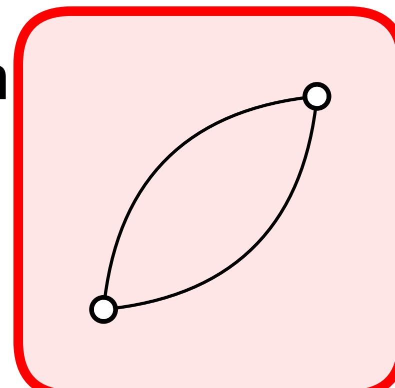
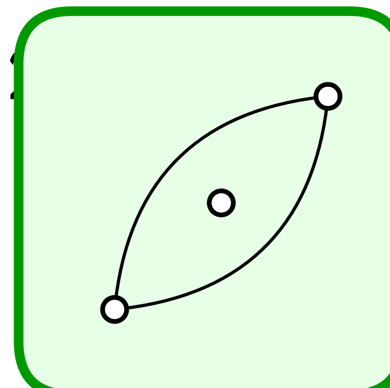
Theorem: min-3-planar graph  $n -$ 

Proof:

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Edge densities: $k = 2$ and $k = 3$

Theorem: min-2-planar graphs have $< 5(n-2)$ edges, which is

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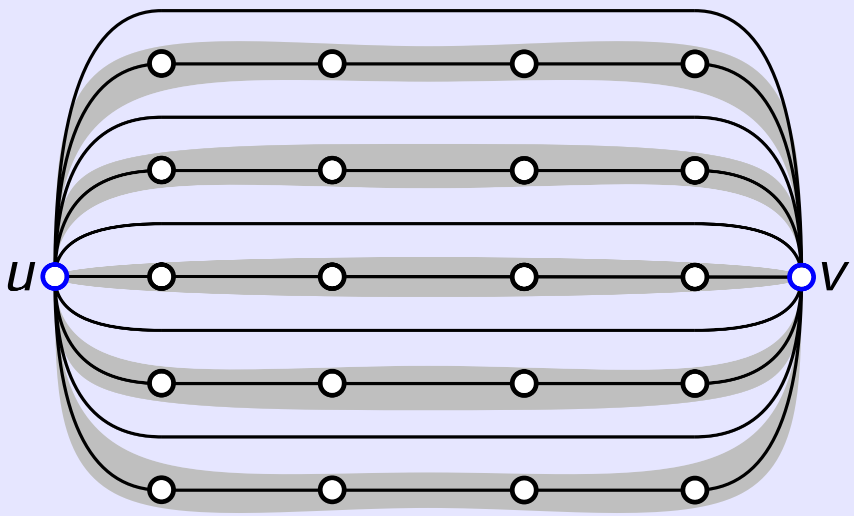

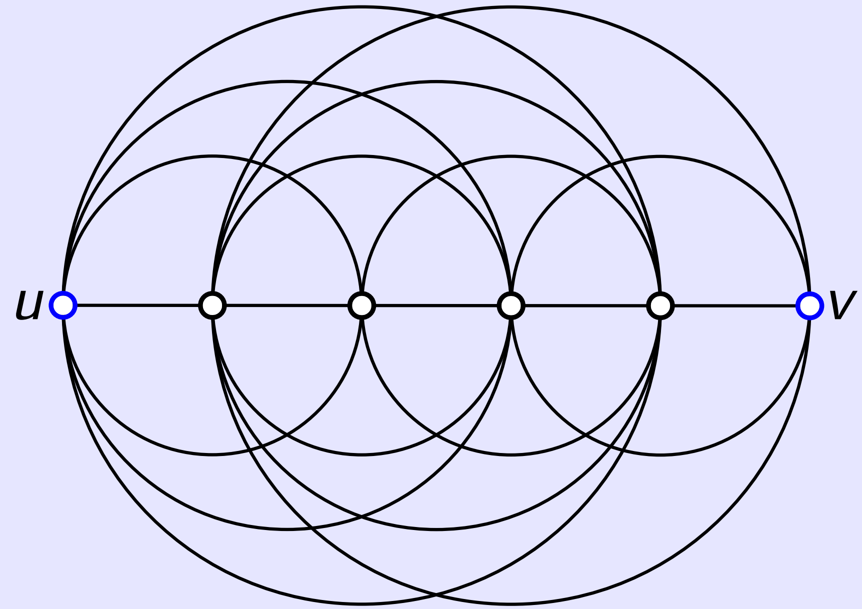

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3-planar:

- Holds also for non-simple graphs (with non-homotopic parallel edges) $\Rightarrow$ Lower bound $5.\overline{6}(n-2)$

Edge densities: $k = 2$ and $k = 3$

Theorem: min-2-planar graphs have $< 5(n-2)$ edges, which is

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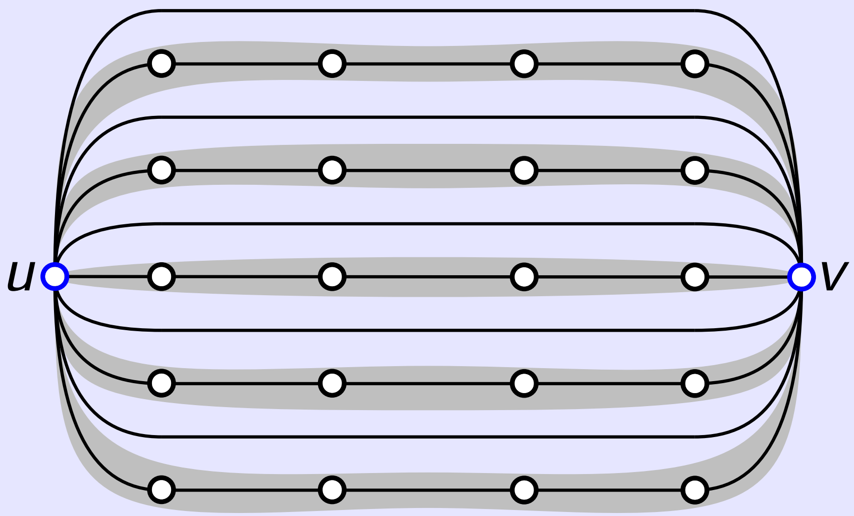

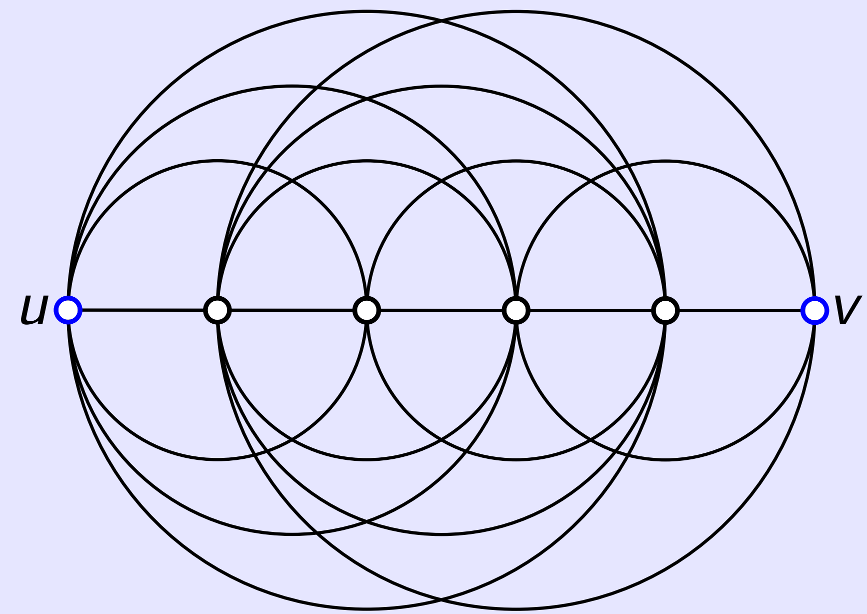

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3-planar:

22 edges / 4 vertices $\Rightarrow 5.5(n-2)$

- Holds also for non-simple graphs (with non-homotopic parallel edges) $\Rightarrow$ Lower bound $5.\overline{6}(n-2)$

Edge densities: $k = 2$ and $k = 3$

Theorem: min-2-planar graphs have $< 5(n-2)$ edges, which is

a

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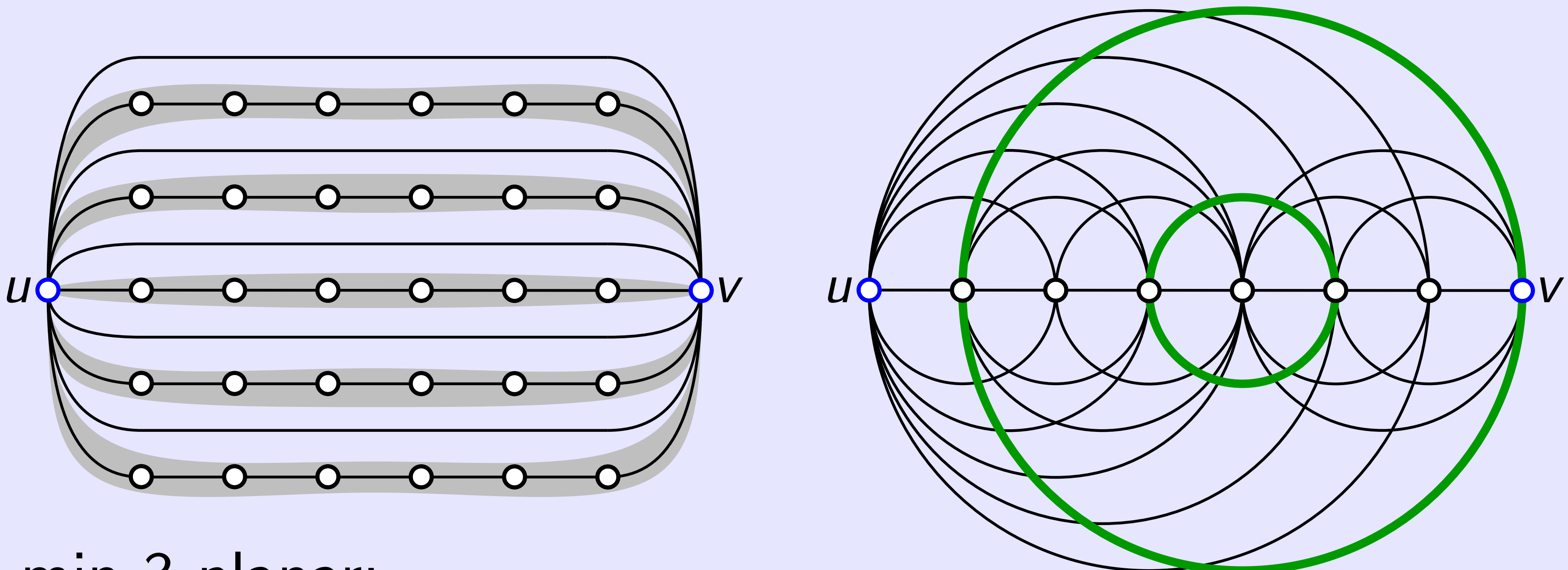

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min-3-planar:

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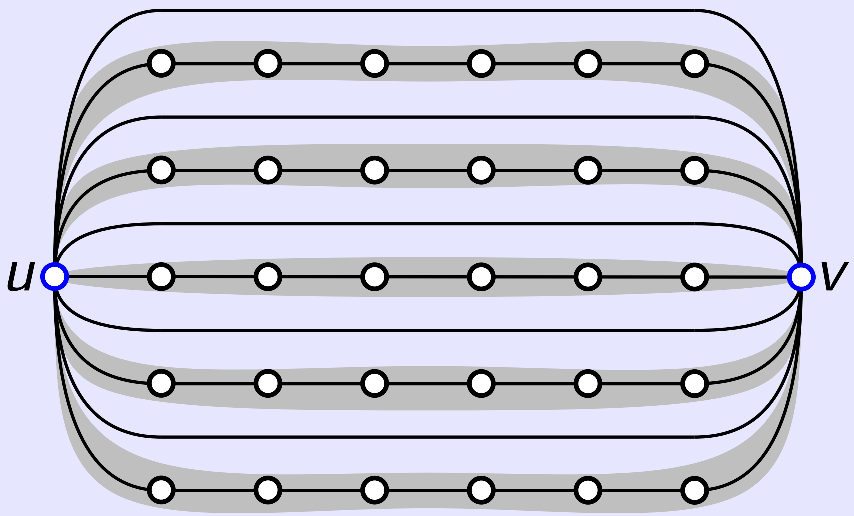

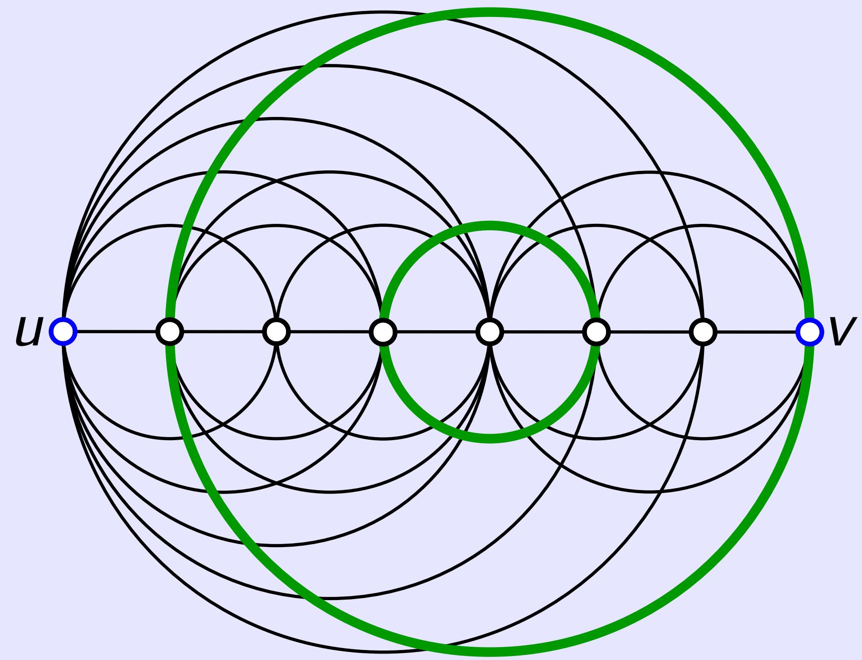

min-3-planar:

34 edges / 6 vertices $\Rightarrow 5.\overline{6}(n-2)$

- Holds also for non-simple graphs (with non-homotopic parallel edges) $\Rightarrow$ Lower bound $5.\overline{6}(n-2)$

Relationships to k-planar graphs

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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- "$\subseteq$" by definition

Relationships to k-planar graphs

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- Let G be planar graph $+$ 1 edge
  $\Rightarrow$ min-1-planar

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

Proof:

- "$\subseteq$" by definition
- Let G be planar graph $+$ 1 edge
  $\Rightarrow$ min-1-planar
  $\Rightarrow$ Not necessary 1-planar [Cabello & Mohar '13]

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Theorem: 2-planar graphs $\subsetneq$ min-2-planar graphs

Relationships to k-planar graphs

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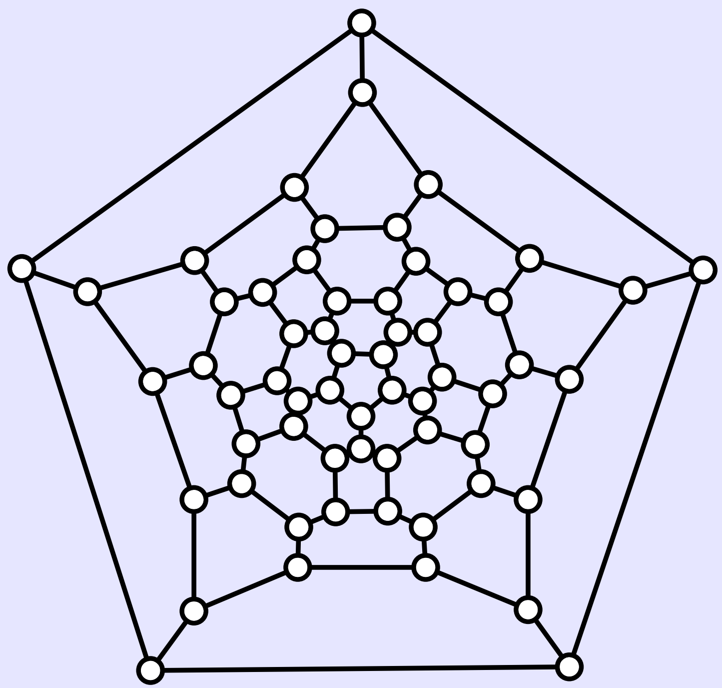
Theorem: 2-planar graphs $\subsetneq$ min-2-planar graphs

Proof:

- "$\subseteq$" by definition
- „Filled" truncated icosahedral graph

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs



planar structure:

$n = 60$

$e = 90$

f: 12 pentagons,

20 hexagons

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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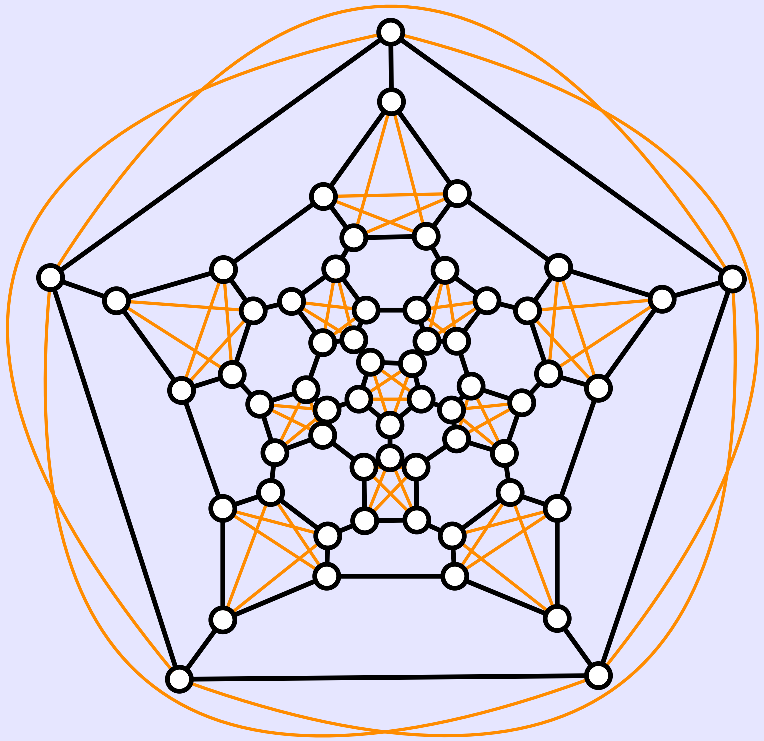

planar structure:

$n = 60$

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f: 12 pentagons, 20 hexagons

fill pentagons by 5 edges

Relationships to *k*-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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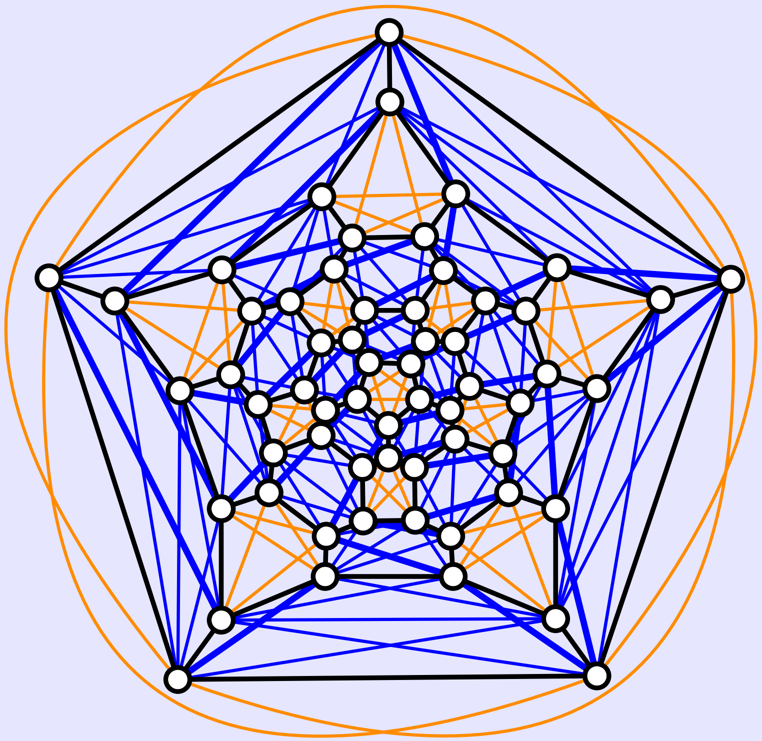

planar structure:

$n = 60$

$e = 90$

f: 12 pentagons,

20 hexagons

fill pentagons by 5 edges

and hexagons by 7 edges

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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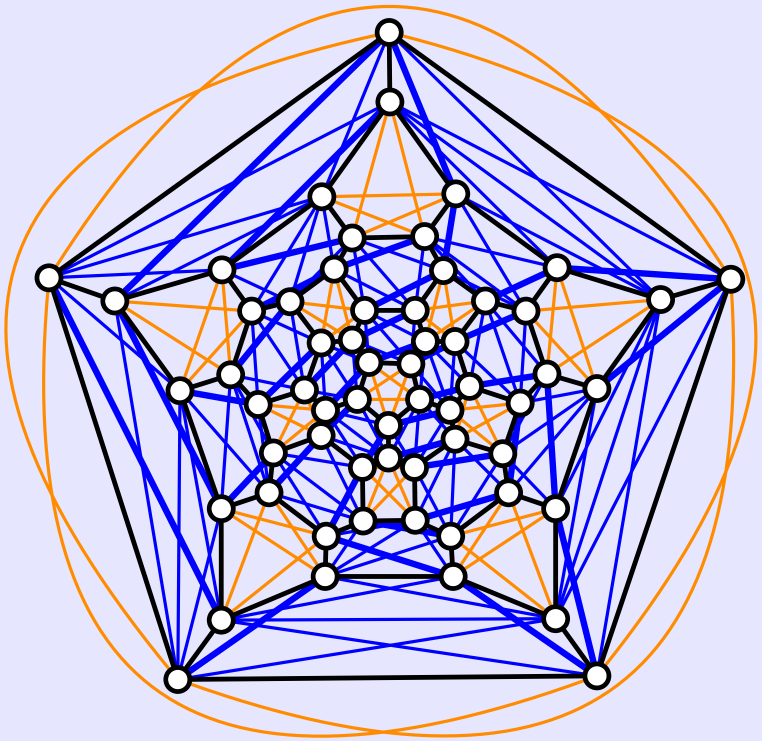

planar structure:

$n = 60$

$e = 90$

f: 12 pentagons, 20 hexagons

fill pentagons by 5 edges and hexagons by 7 edges

$\Rightarrow 90 + 12 \cdot 5 + 20 \cdot 7 = 290$ edges

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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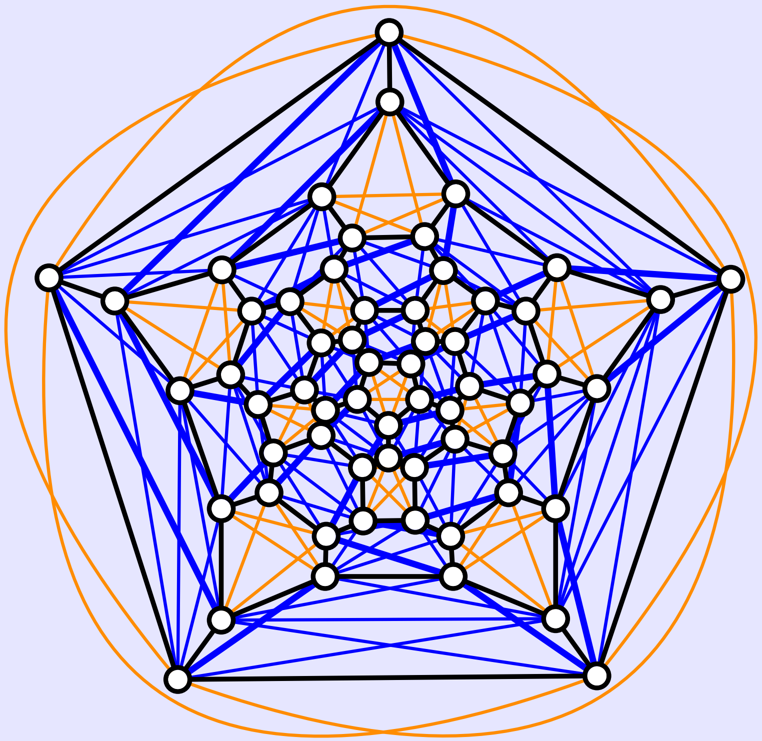

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Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

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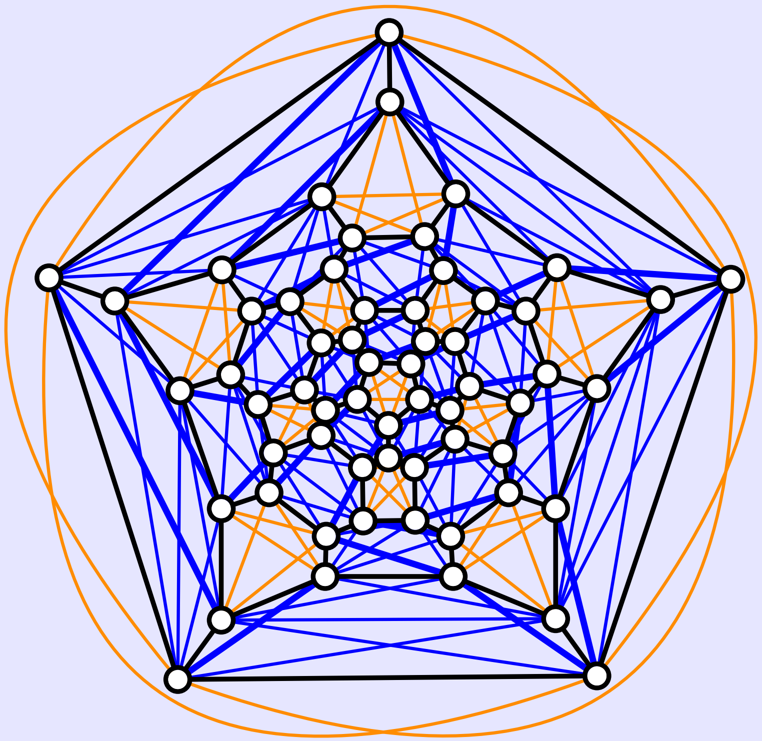

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$\Rightarrow 90 + 12 \cdot 5 + 20 \cdot 7 = 290$ edges $\Rightarrow = 5(n-2)$

$\Rightarrow$ but not optimal 2-planar [Förster et al., '21]

Relationships to k-planar graphs

Theorem: 1-planar graphs $\subsetneq$ min-1-planar graphs

Proof:

- "$\subseteq$" by definition
- Let G be planar graph $+ 1$ edge
  $\Rightarrow$ min-1-planar
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Relationships to k-planar graphs

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Theorem: 2-planar graphs $\subsetneq$ min-2-planar graphs

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- "$\subseteq$" by definition
- „Filled" truncated icosahedral graph
- Holds also for optimal 2-planar / min-2-planar graphs

Open problems

Open problems

- Better bounds for edge densities

Open problems

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upper bounds:

k	k-planar	min-k-planar
1	$4(n-2)$	$4(n-2)$
2	$5(n-2)$	$5(n-2)$
3	$5.5(n-2)$	$6(n-2)$
4	$6(n-2)$	$(3.81\sqrt{k}+3)n$
≥ 5	$3.81\sqrt{k} \cdot n$	

Open problems

- Better bounds for edge densities
  - Seems hard for $k = 3$

upper bounds:

<table>
<tr><th>k</th><th>k-planar</th><th>min-k-planar</th></tr>
<tr><td>1</td><td>$4(n-2)$</td><td>$4(n-2)$</td></tr>
<tr><td>2</td><td>$5(n-2)$</td><td>$5(n-2)$</td></tr>
<tr><td>3</td><td>$5.5(n-2)$</td><td>$6(n-2)$</td></tr>
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</table>

Open problems

- Better bounds for edge densities
  - Seems hard for $k = 3$
  - Different to k-planar for greater k?

upper bounds:

k	k-planar	min-k-planar
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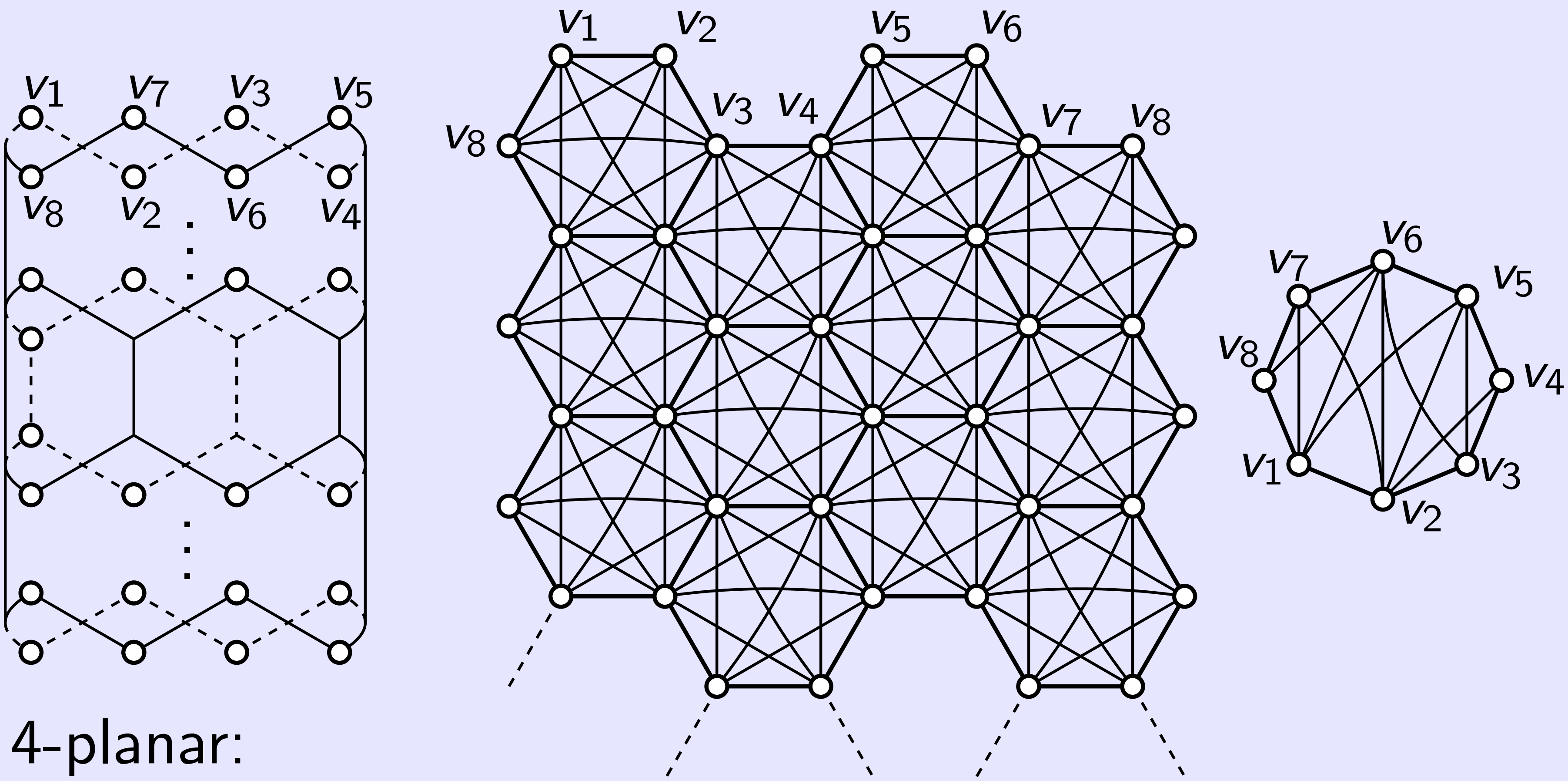
Open problems

- Better bounds for edge densities

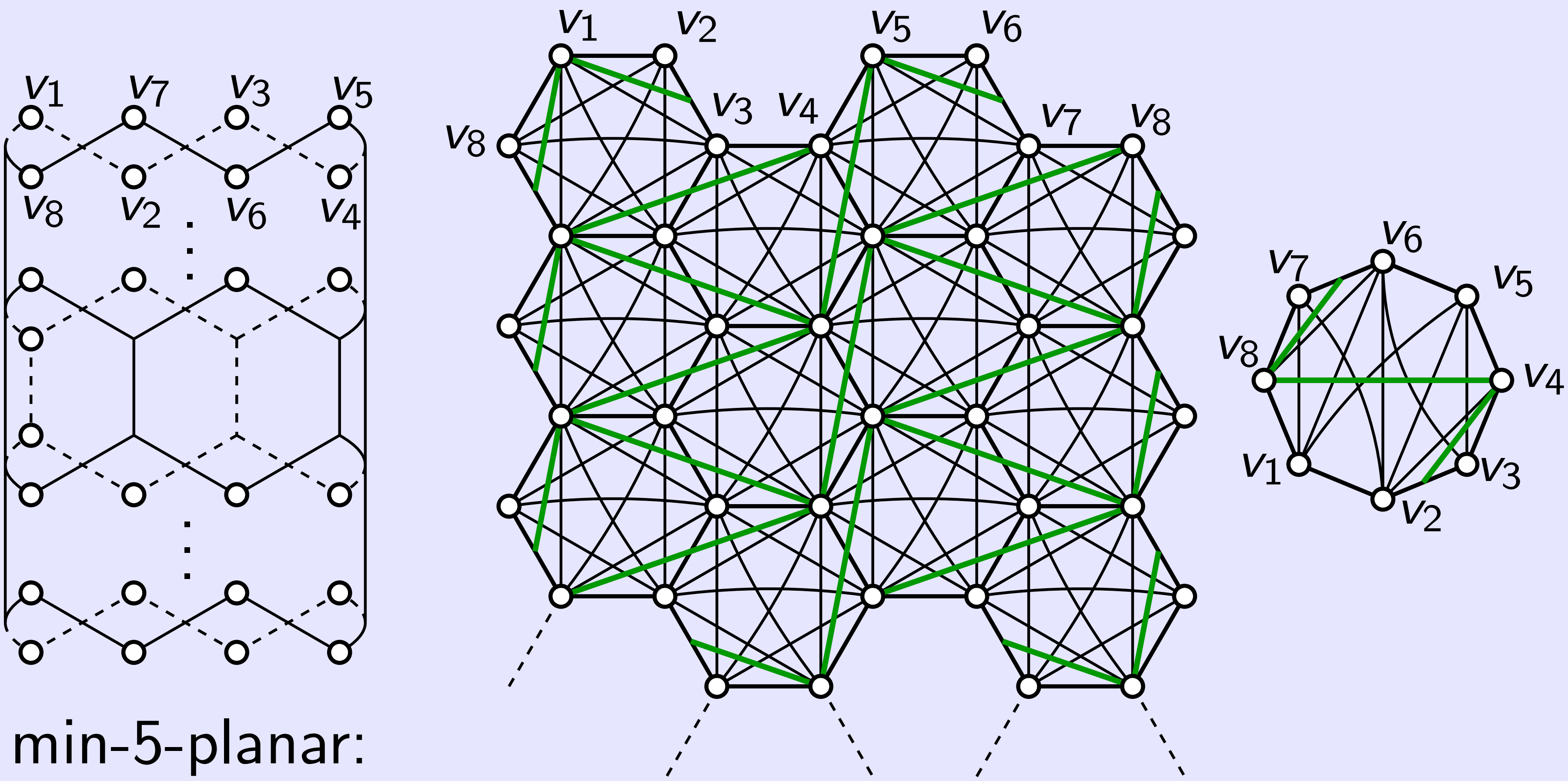


4-planar:

$6n - O(1)$ edges

Open problems

- Better bounds for edge densities



min-5-planar:

$6n + 0.75n - O(1) = 6.75n - O(1)$ edges

Open problems

- Better bounds for edge densities
  - Seems hard for $k = 3$
  - Different to k-planar for greater k?

upper bounds:

k	k-planar	min-k-planar
1	$4(n-2)$	$4(n-2)$
2	$5(n-2)$	$5(n-2)$
3	$5.5(n-2)$	$6(n-2)$
4	$6(n-2)$	$(3.81\sqrt{k}+3)n$
≥ 5	$3.81\sqrt{k} \cdot n$	

Open problems

- Better bounds for edge densities
  - Seems hard for $k = 3$
  - Different to k-planar for greater k?
- Relationships to other graph classes

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- Better bounds for edge densities
  - Seems hard for $k = 3$
  - Different to k-planar for greater k?
- Relationships to other graph classes
- Recognition, ...

Open problems

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  - Seems hard for $k = 3$
  - Different to k-planar for greater k?
- Relationships to other graph classes
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Thank You!