String graphs with precise number of intersections

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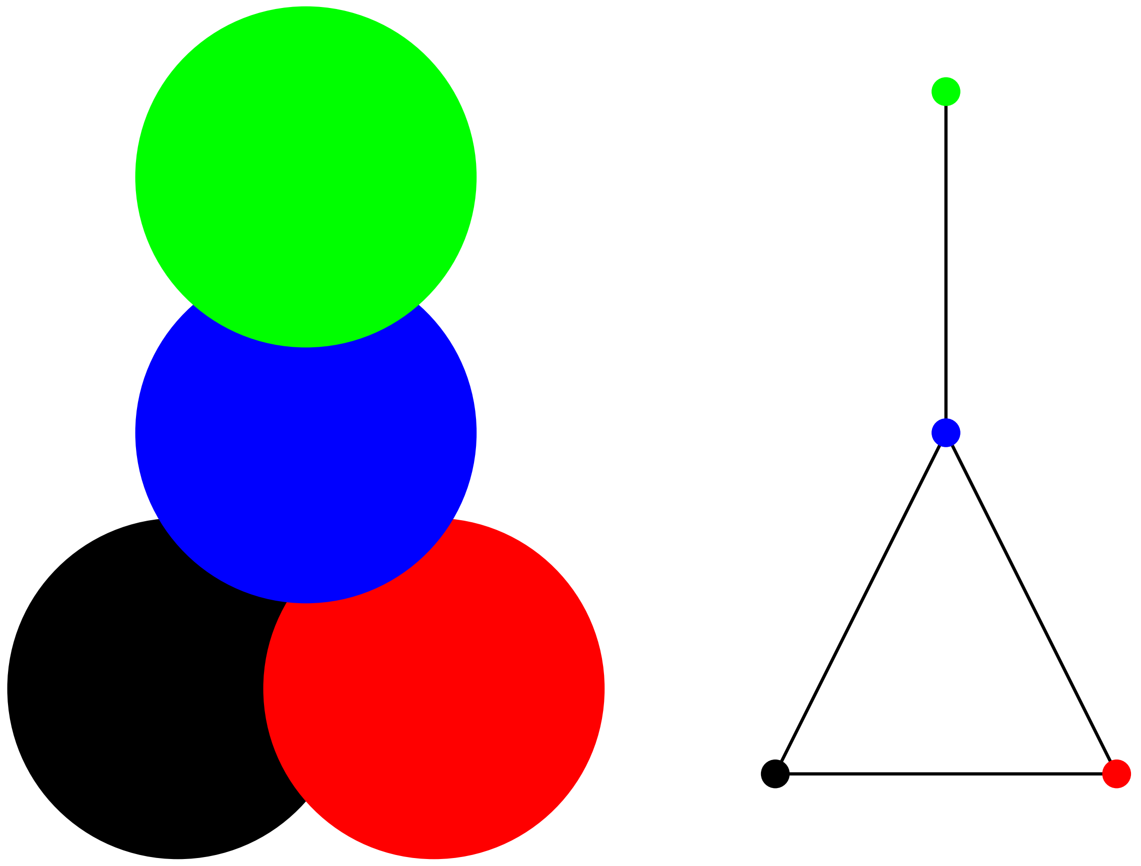
Charles University, Prague

Introduction

- Intersection graphs
  - vertices $\sim$ objects
  - edges $\sim$ nonempty intersections



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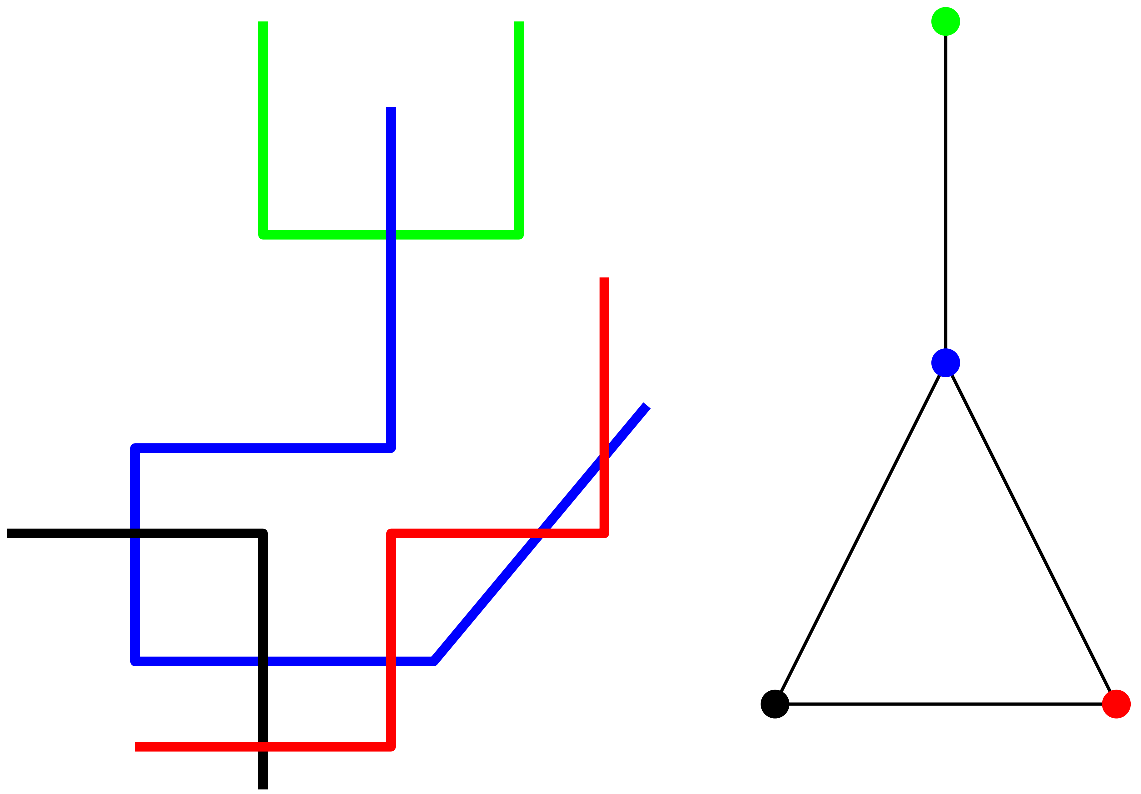

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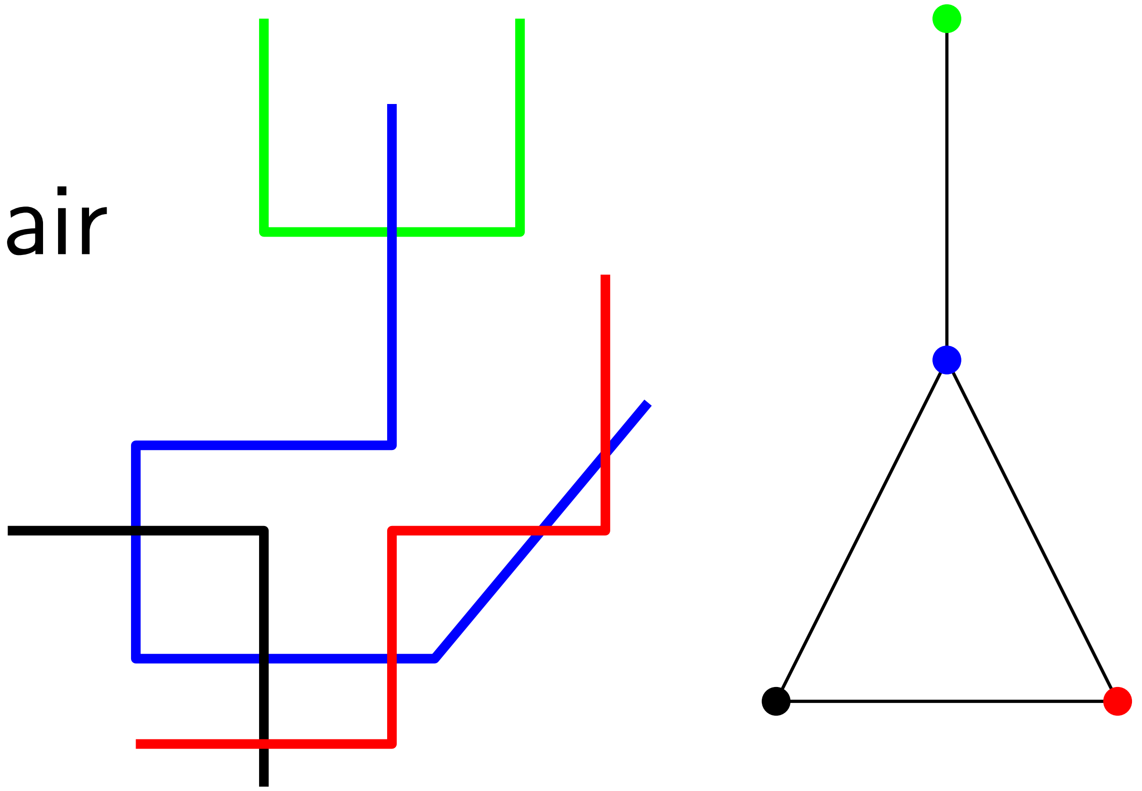

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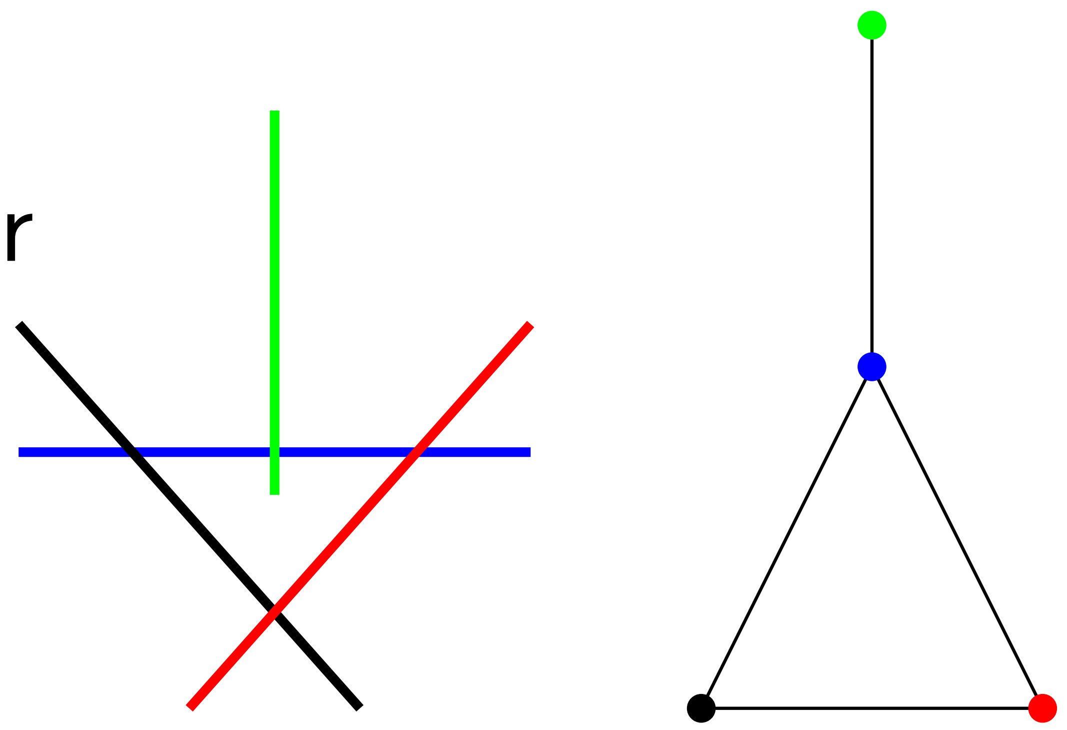

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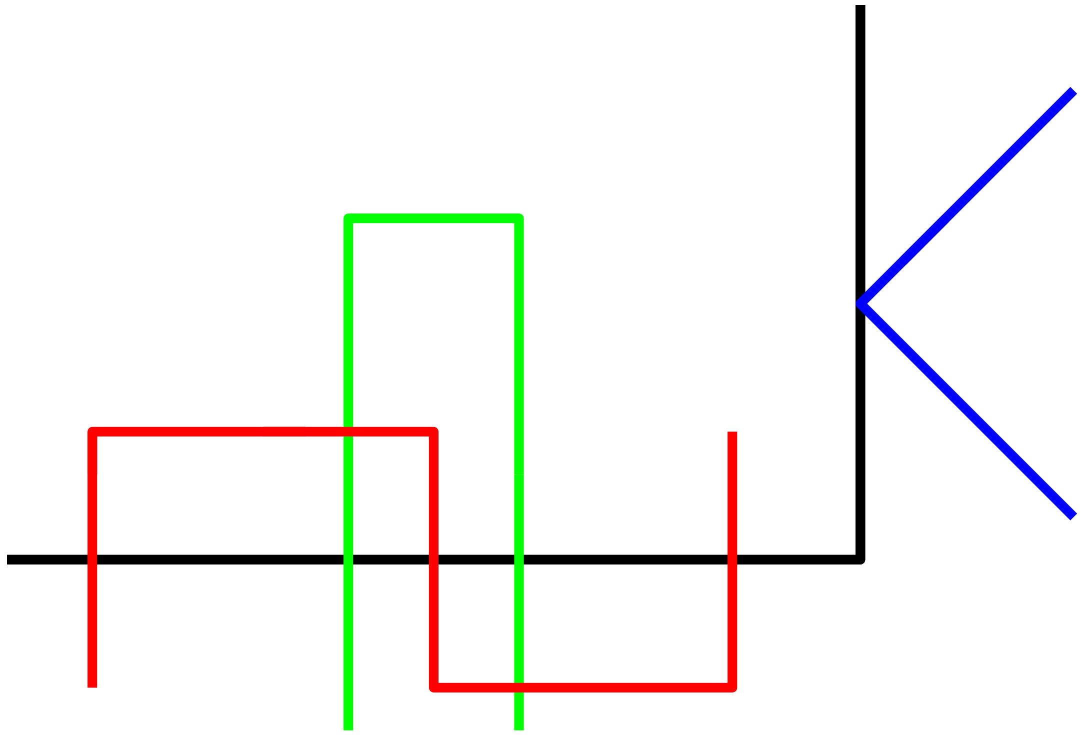

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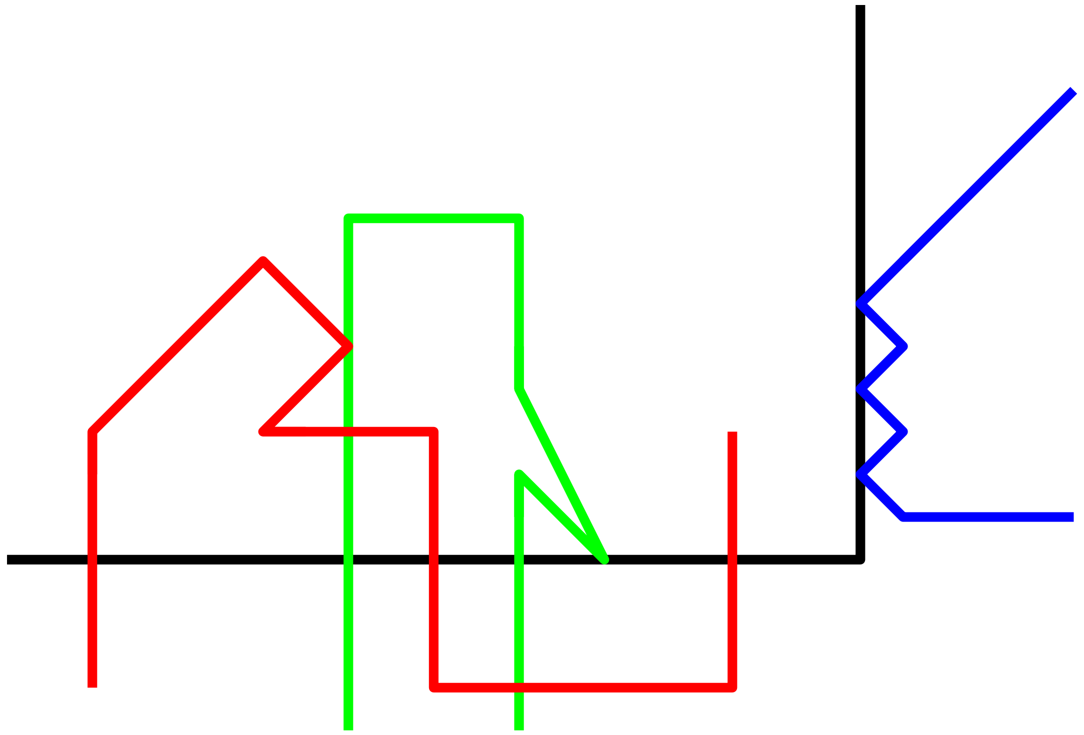

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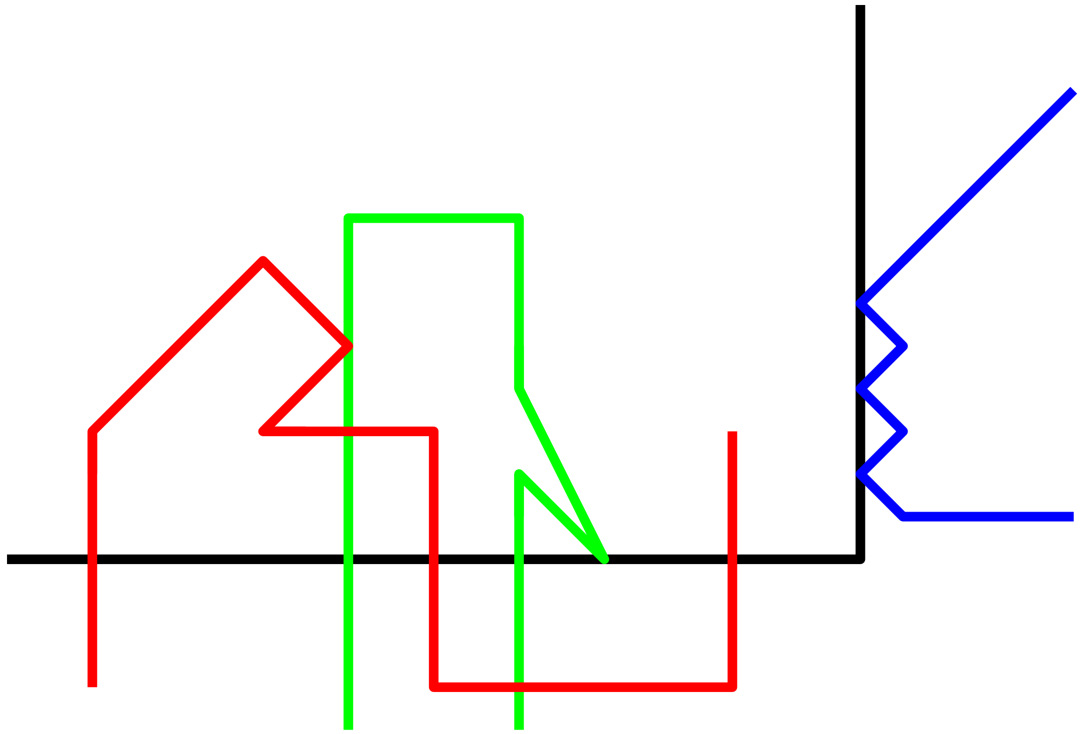

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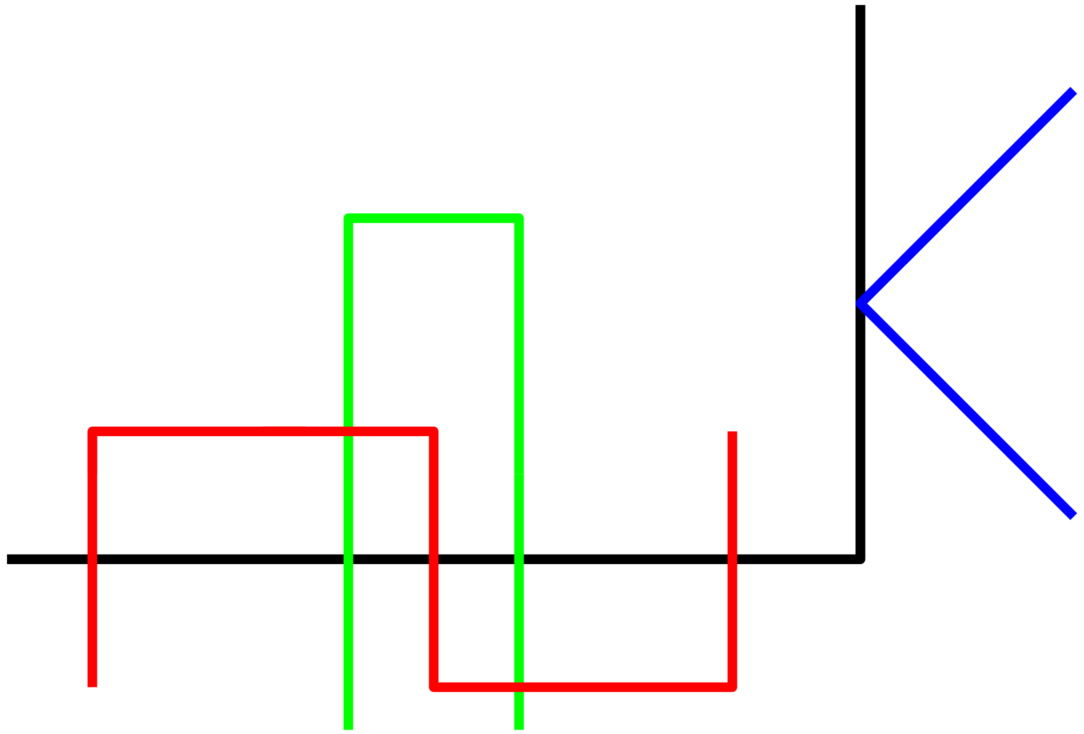

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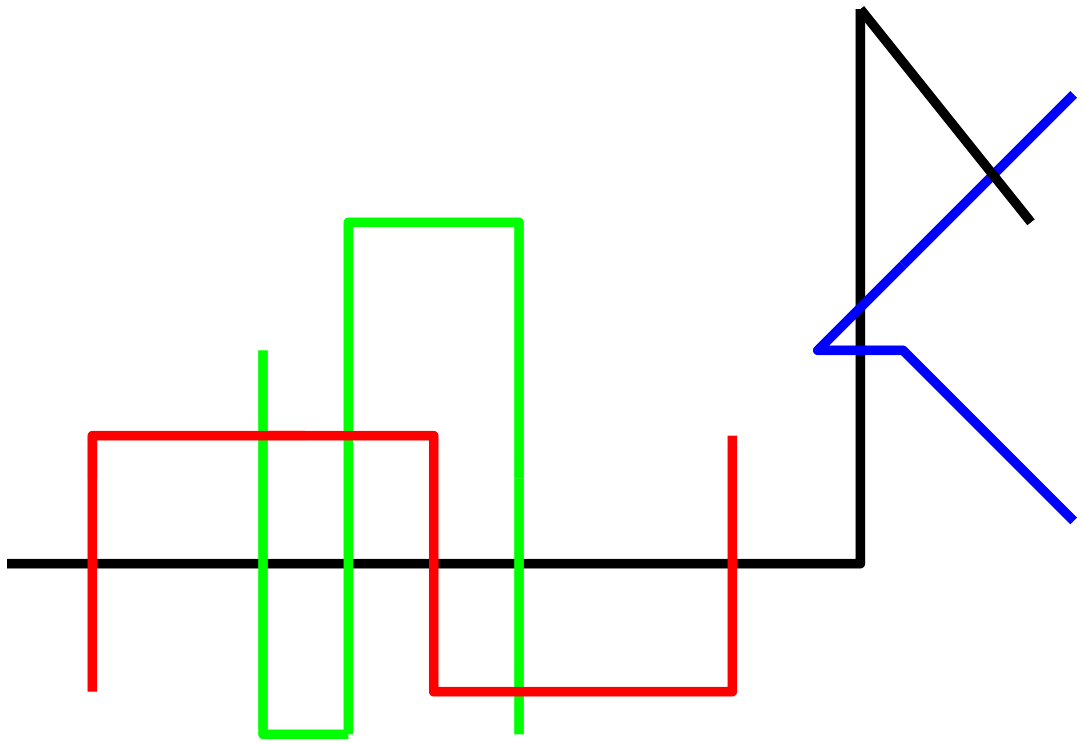

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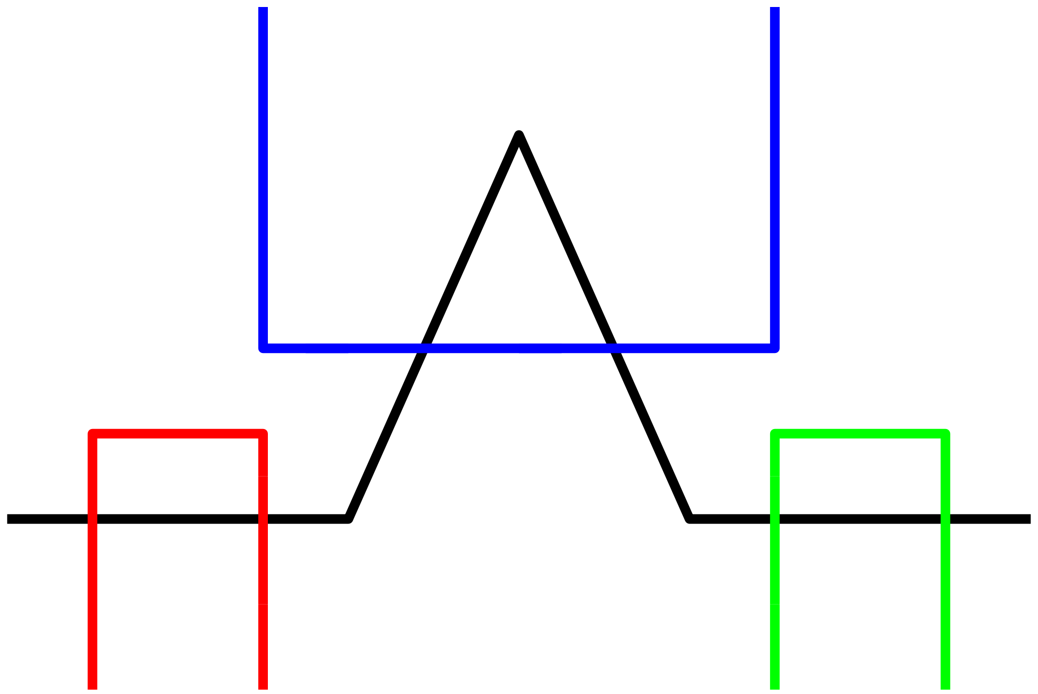

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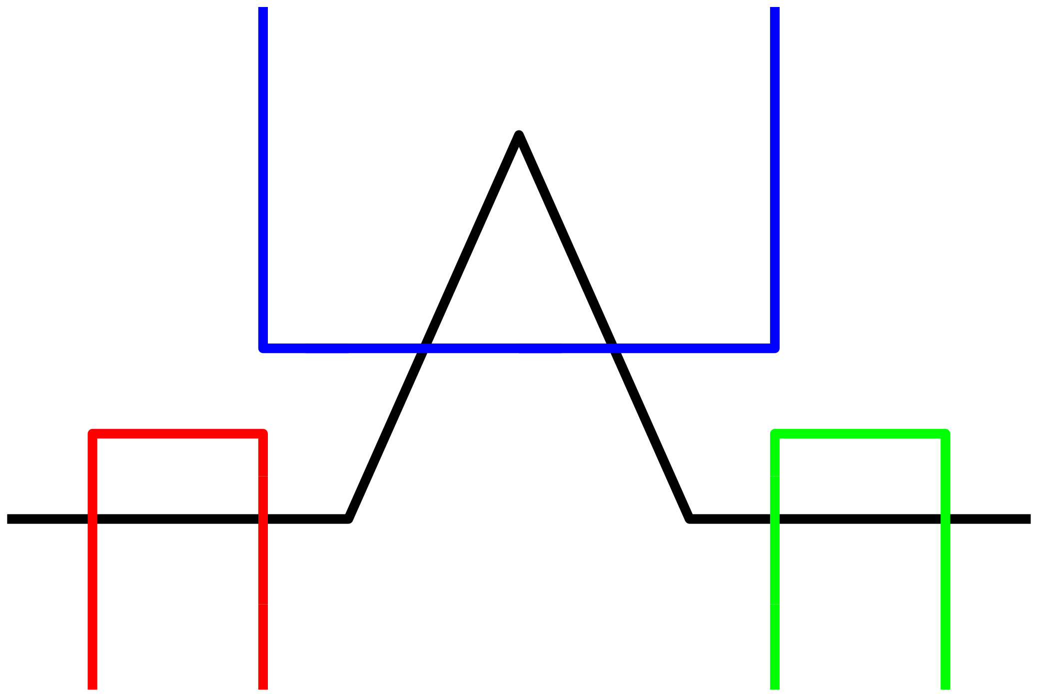

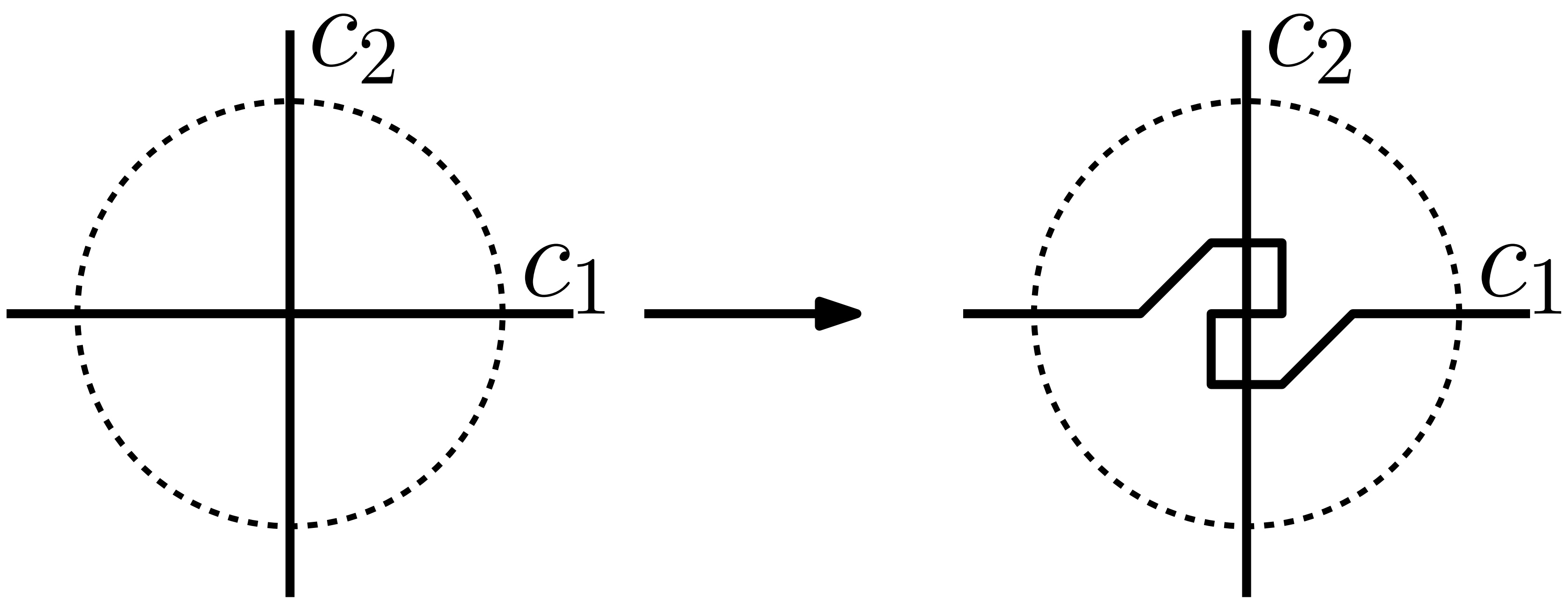

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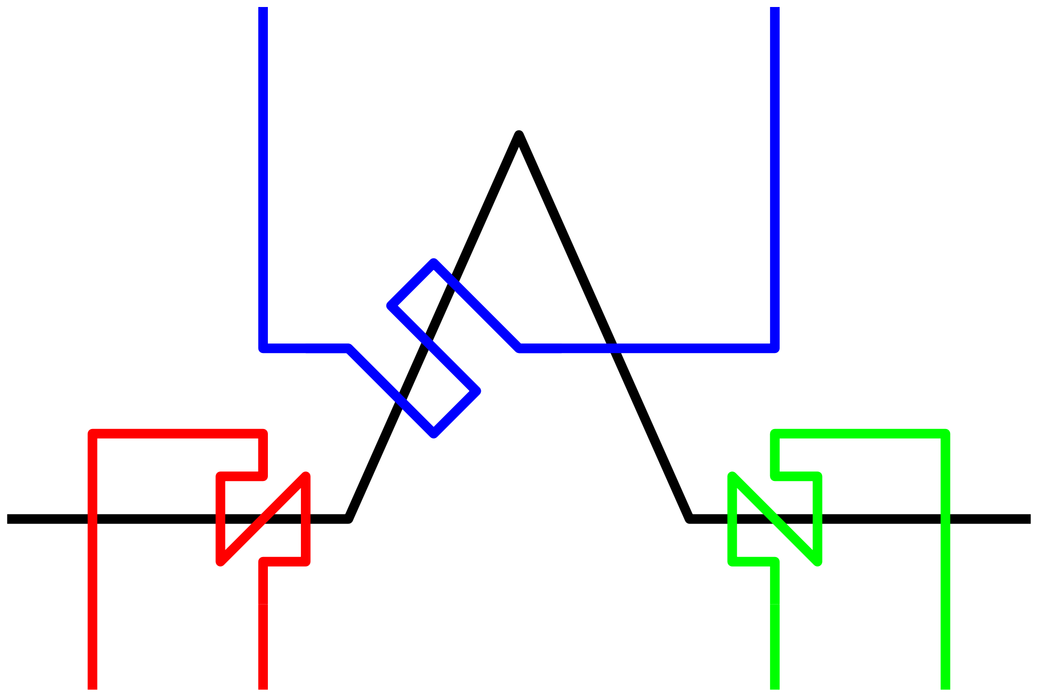

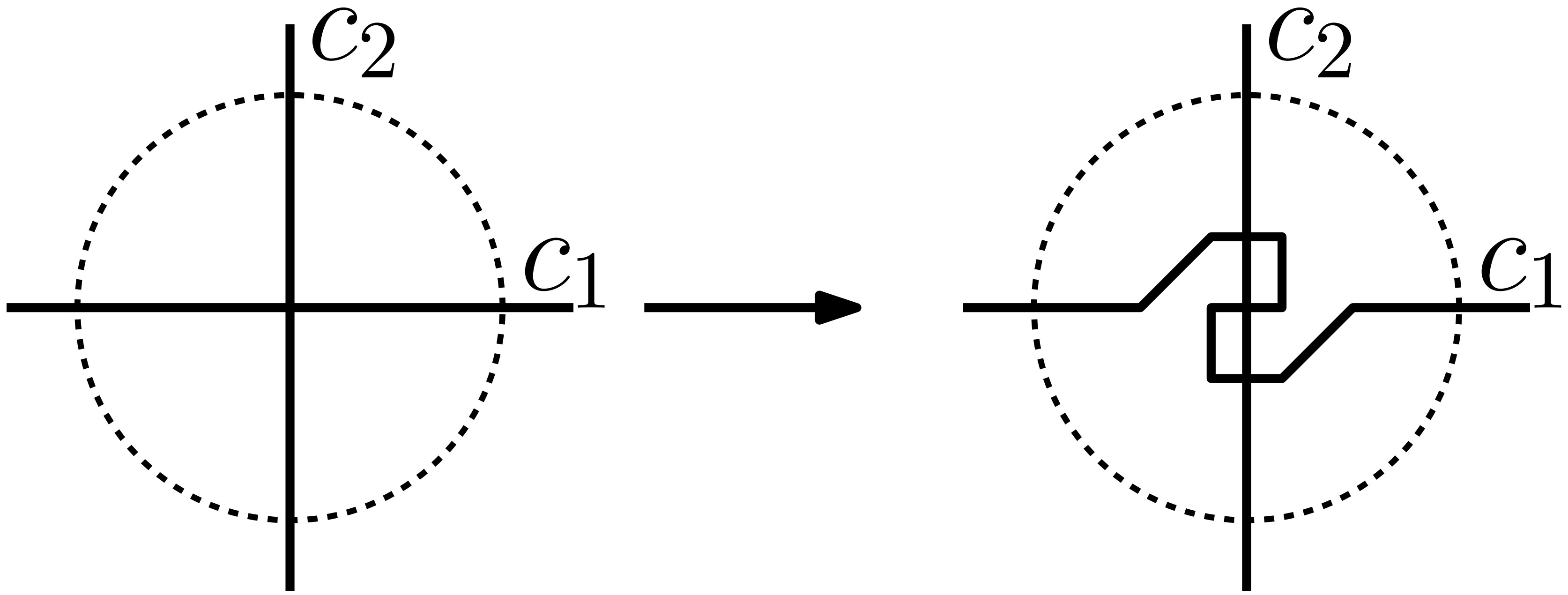

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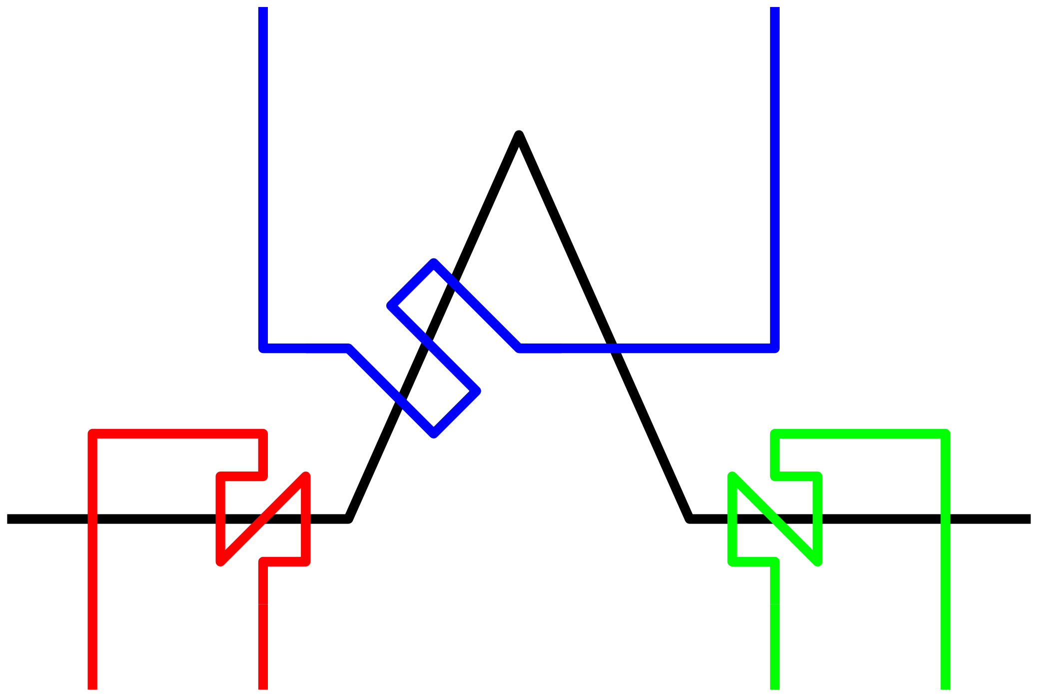

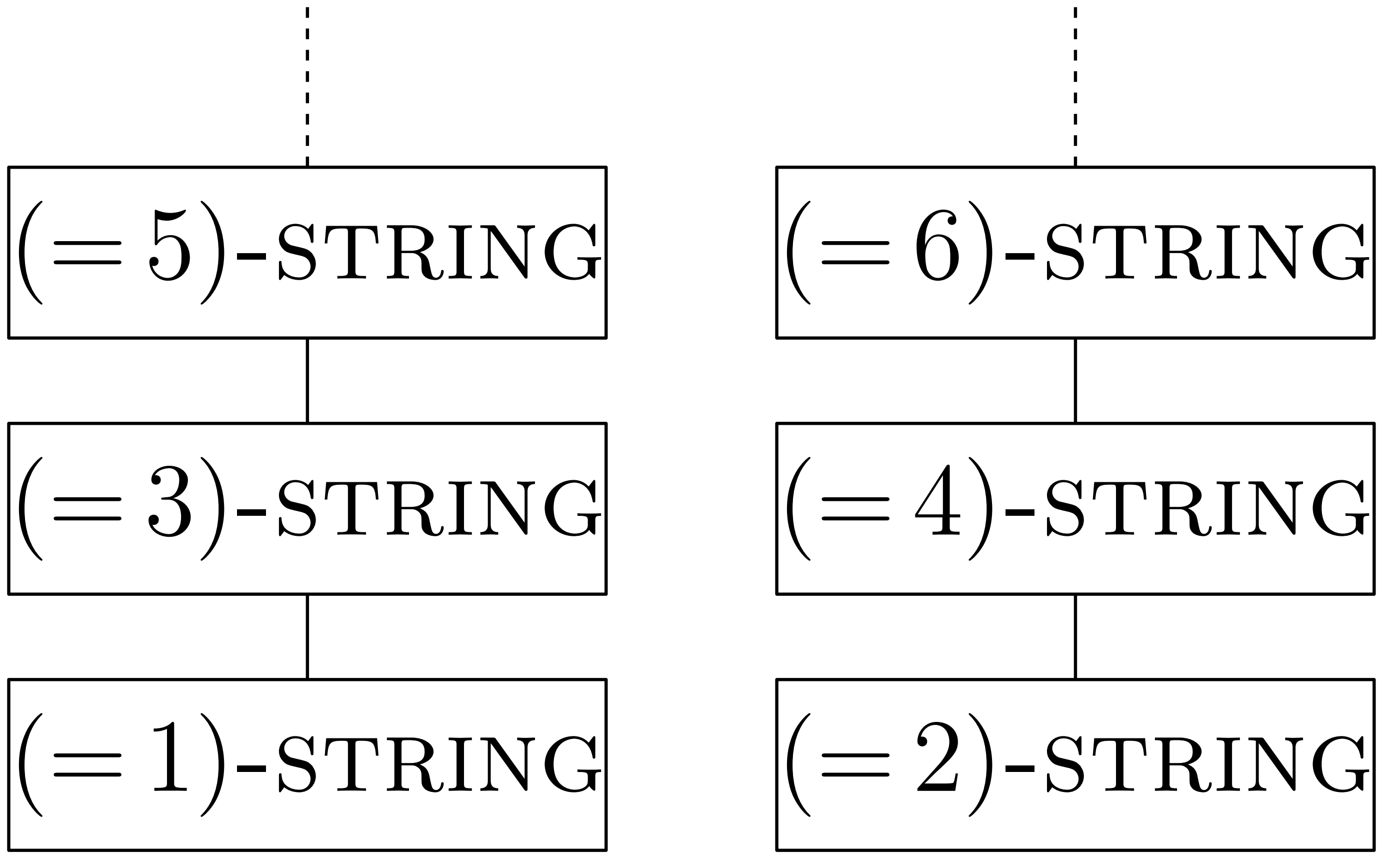

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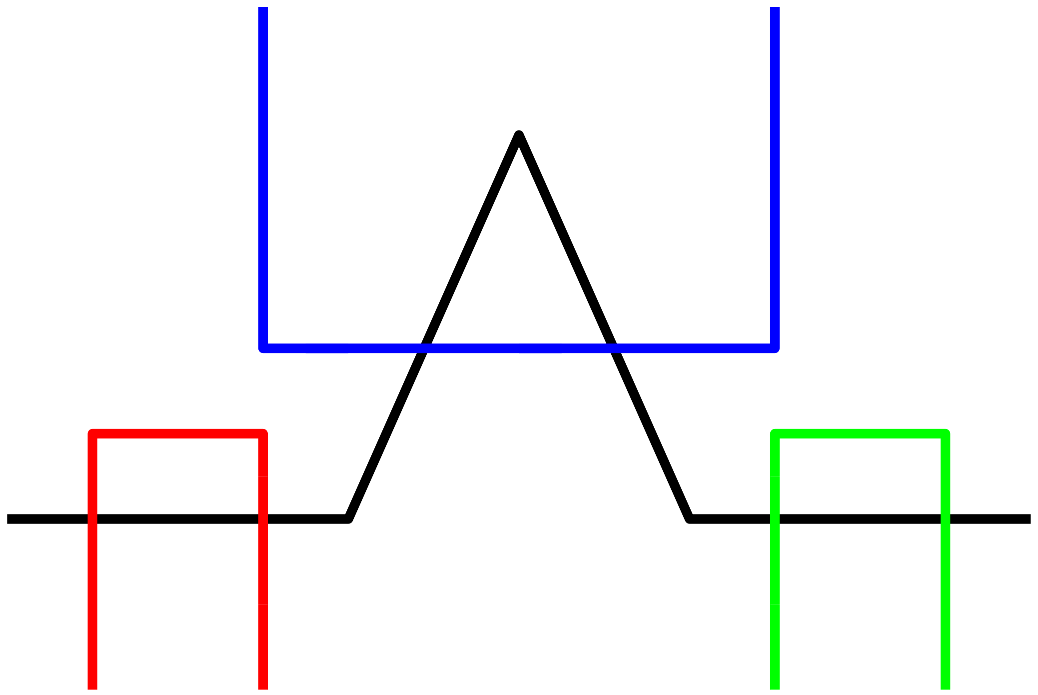

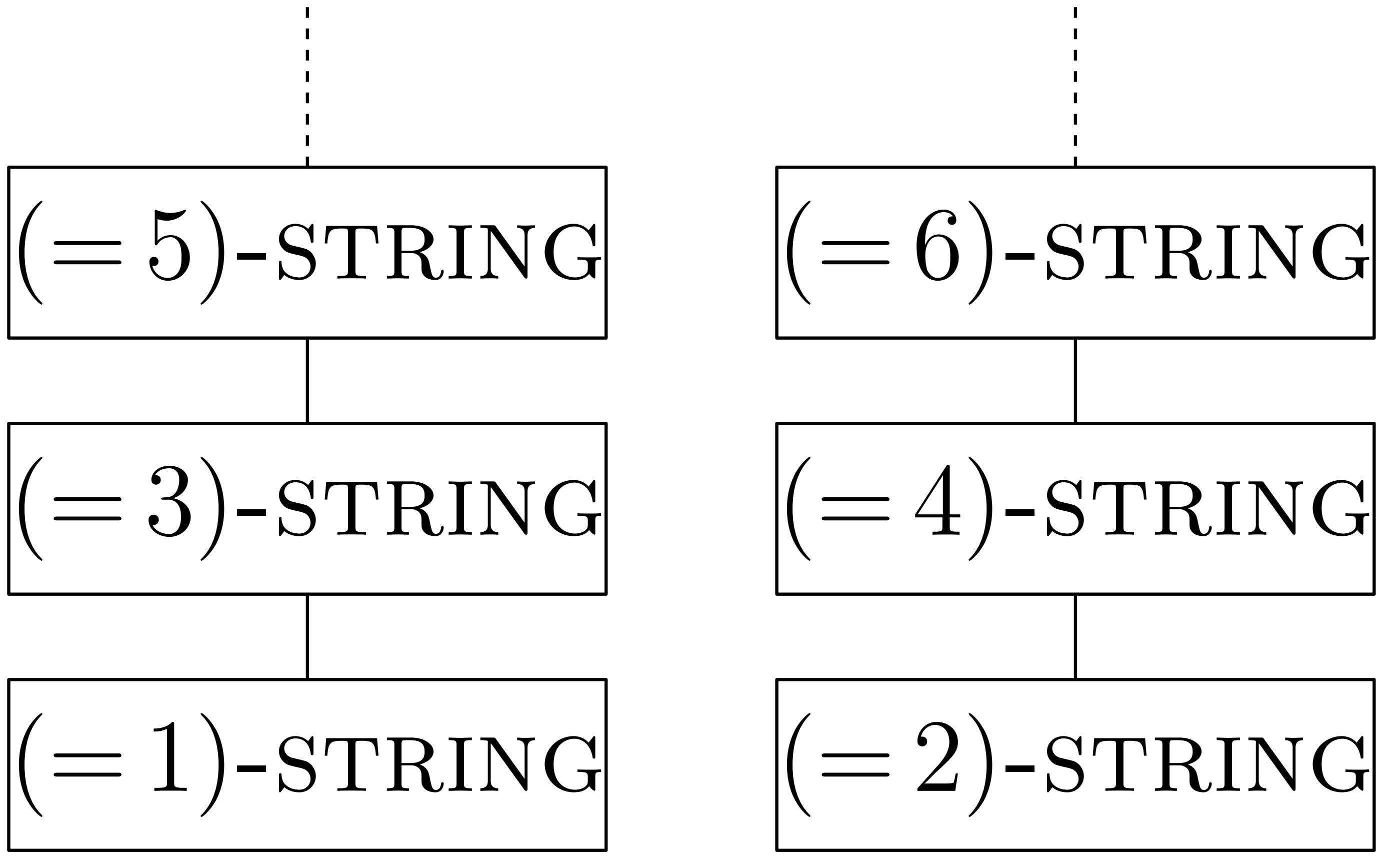

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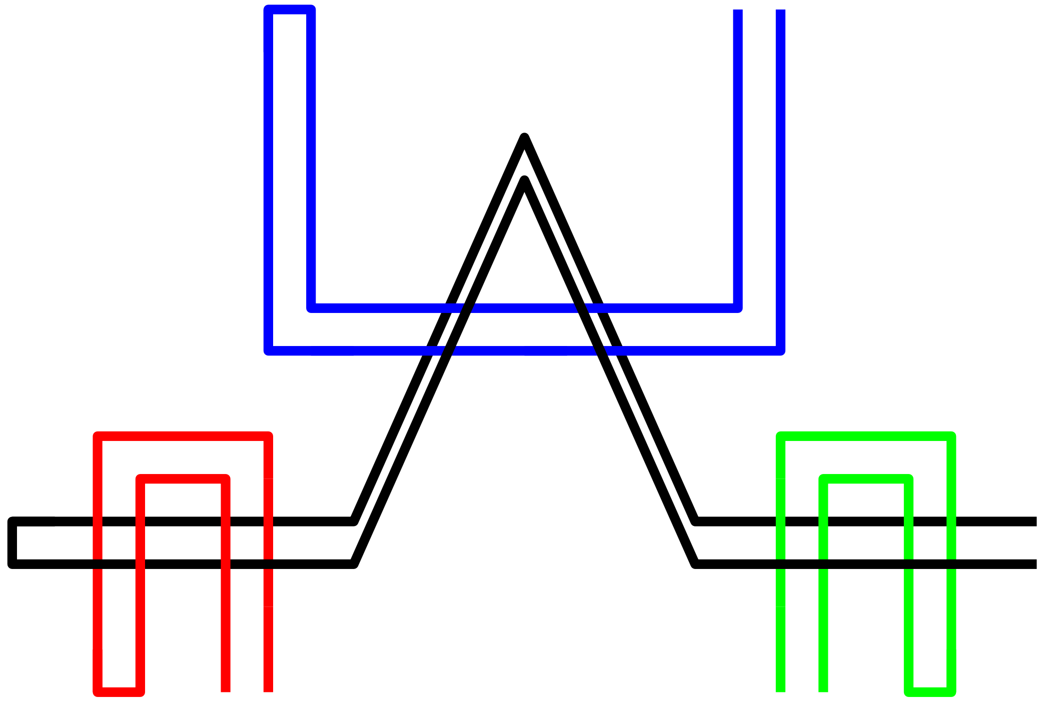

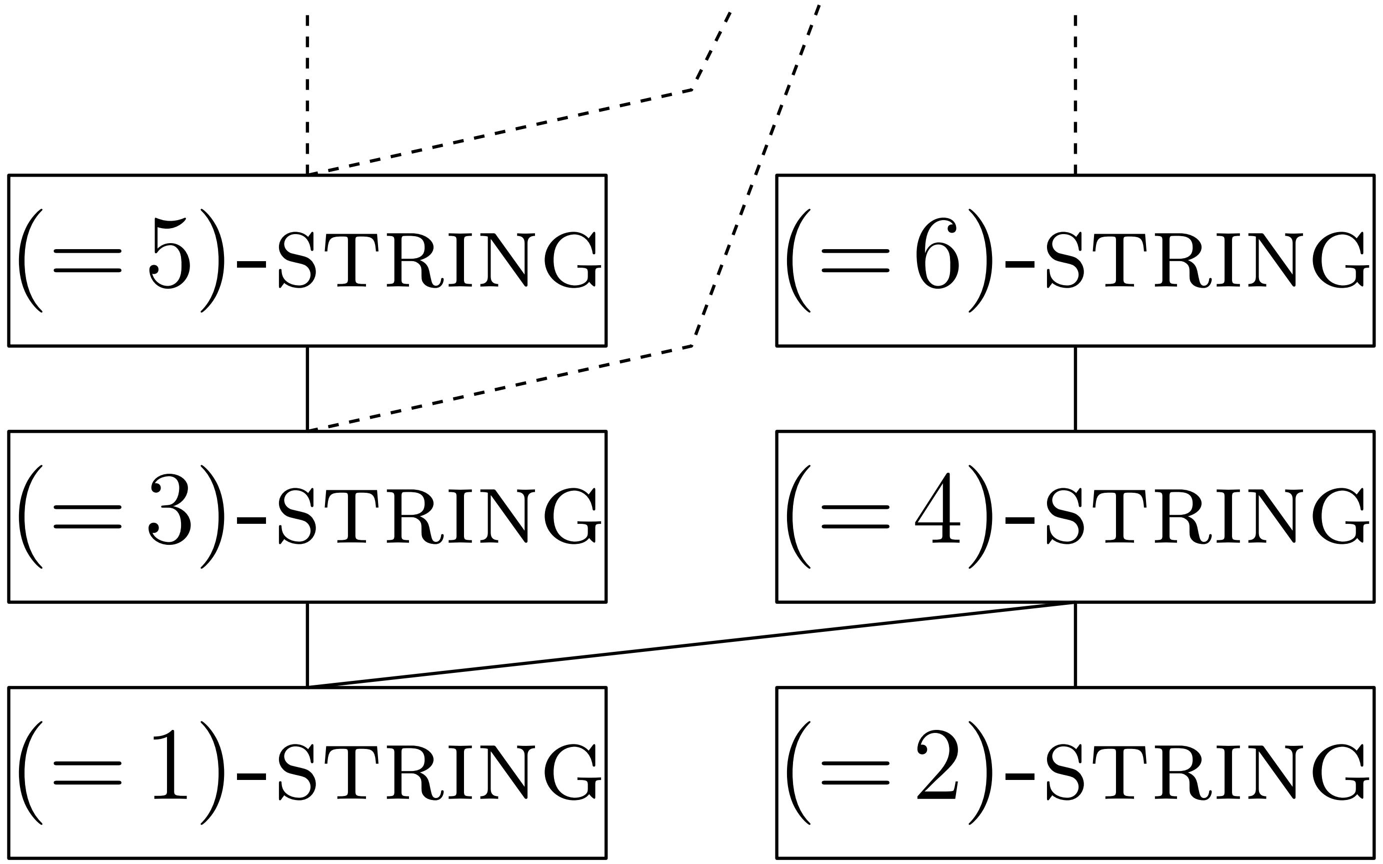

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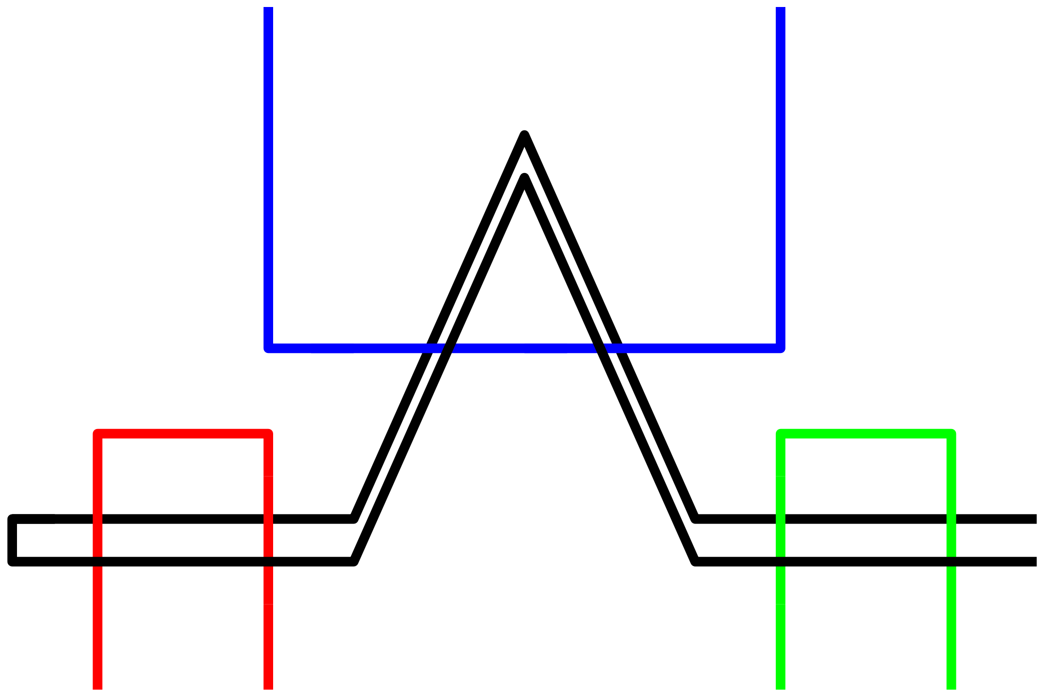

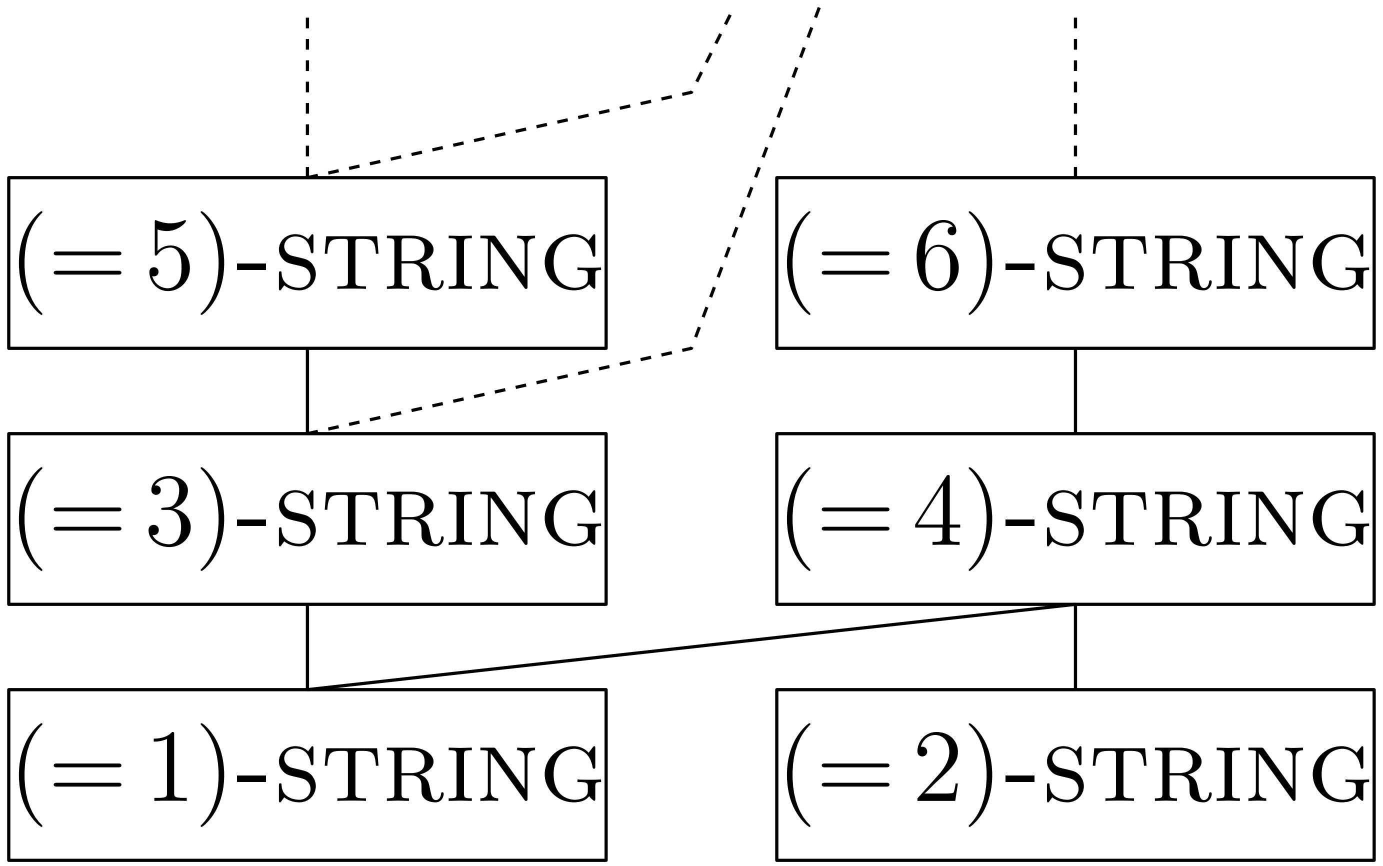

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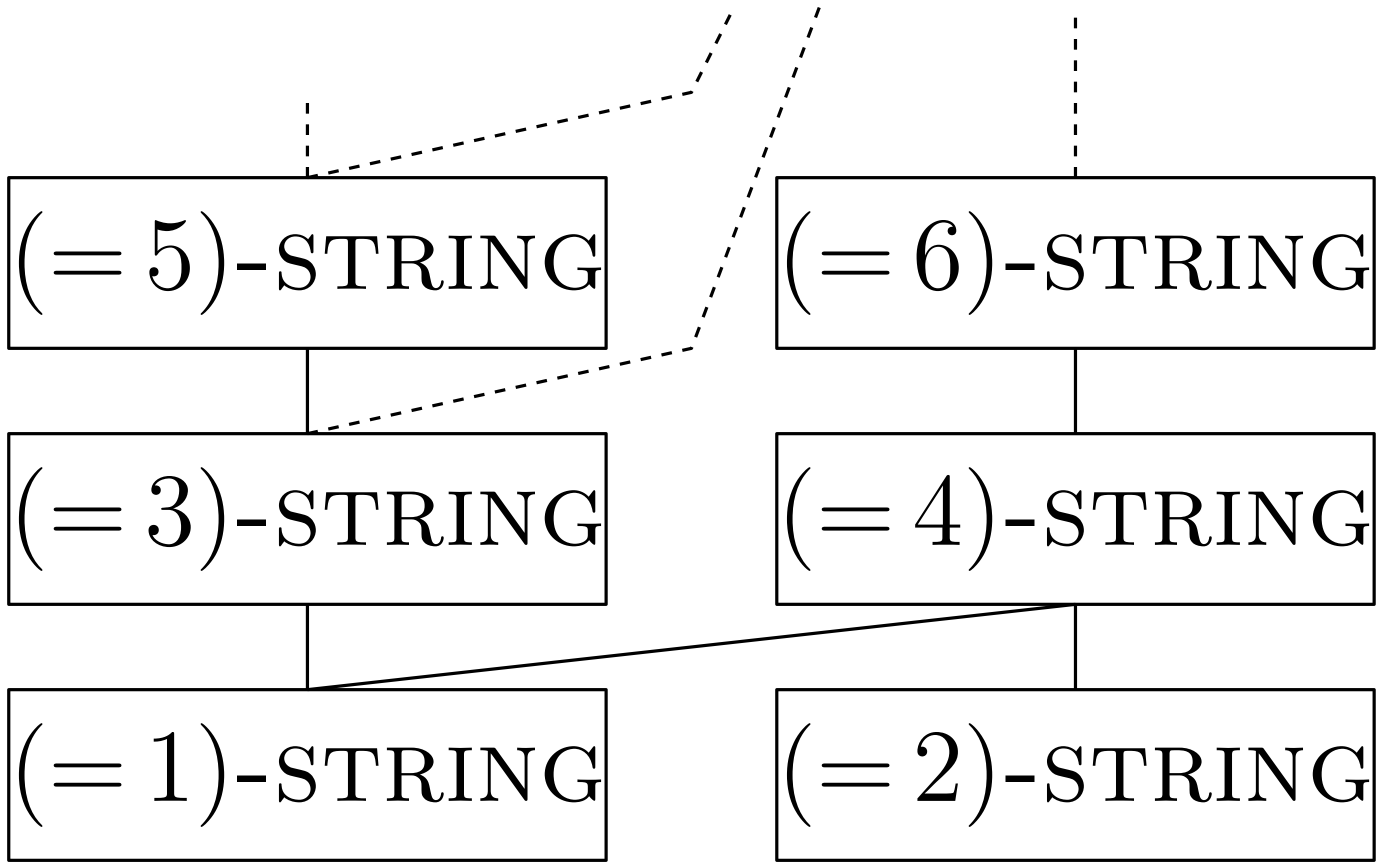

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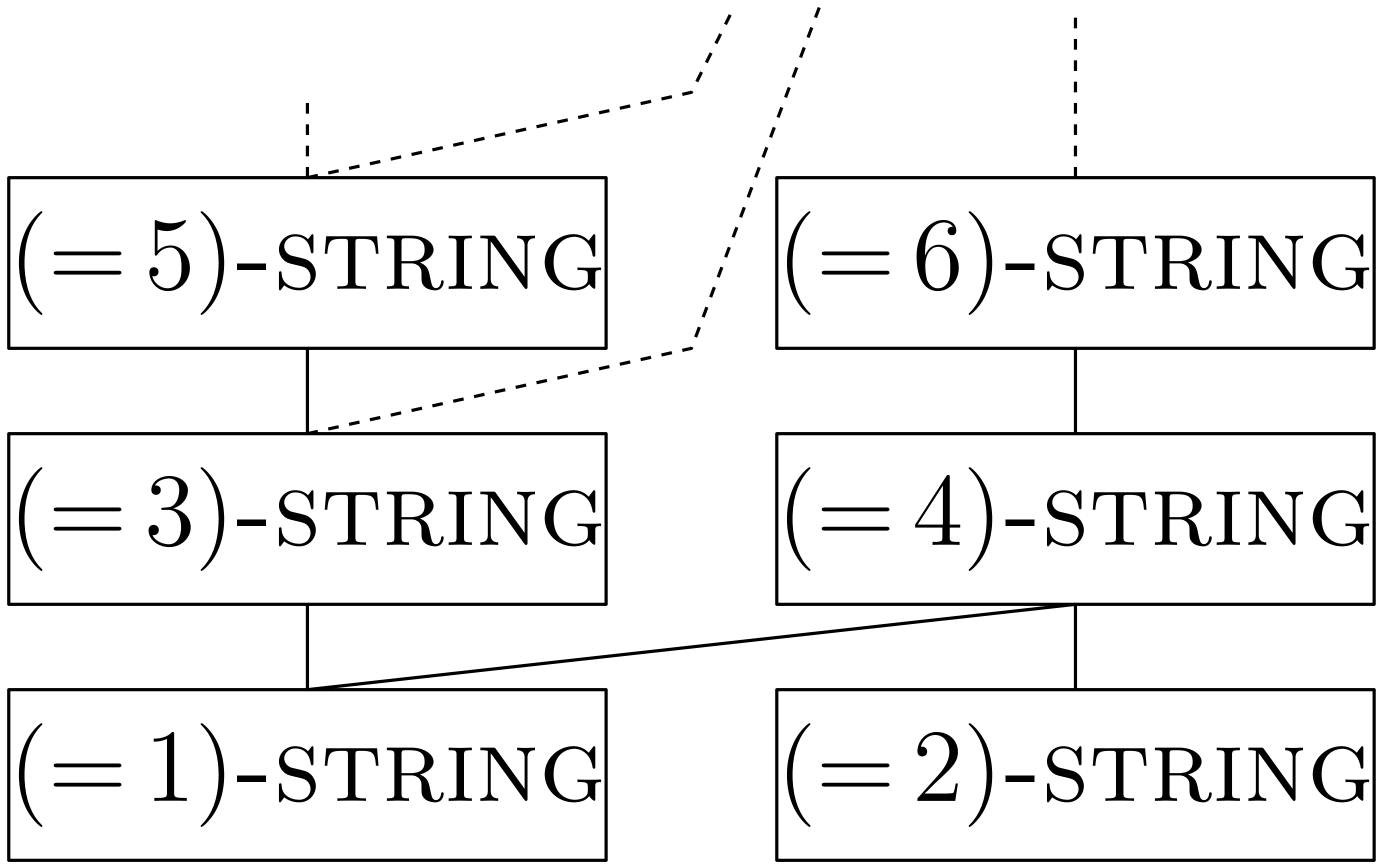

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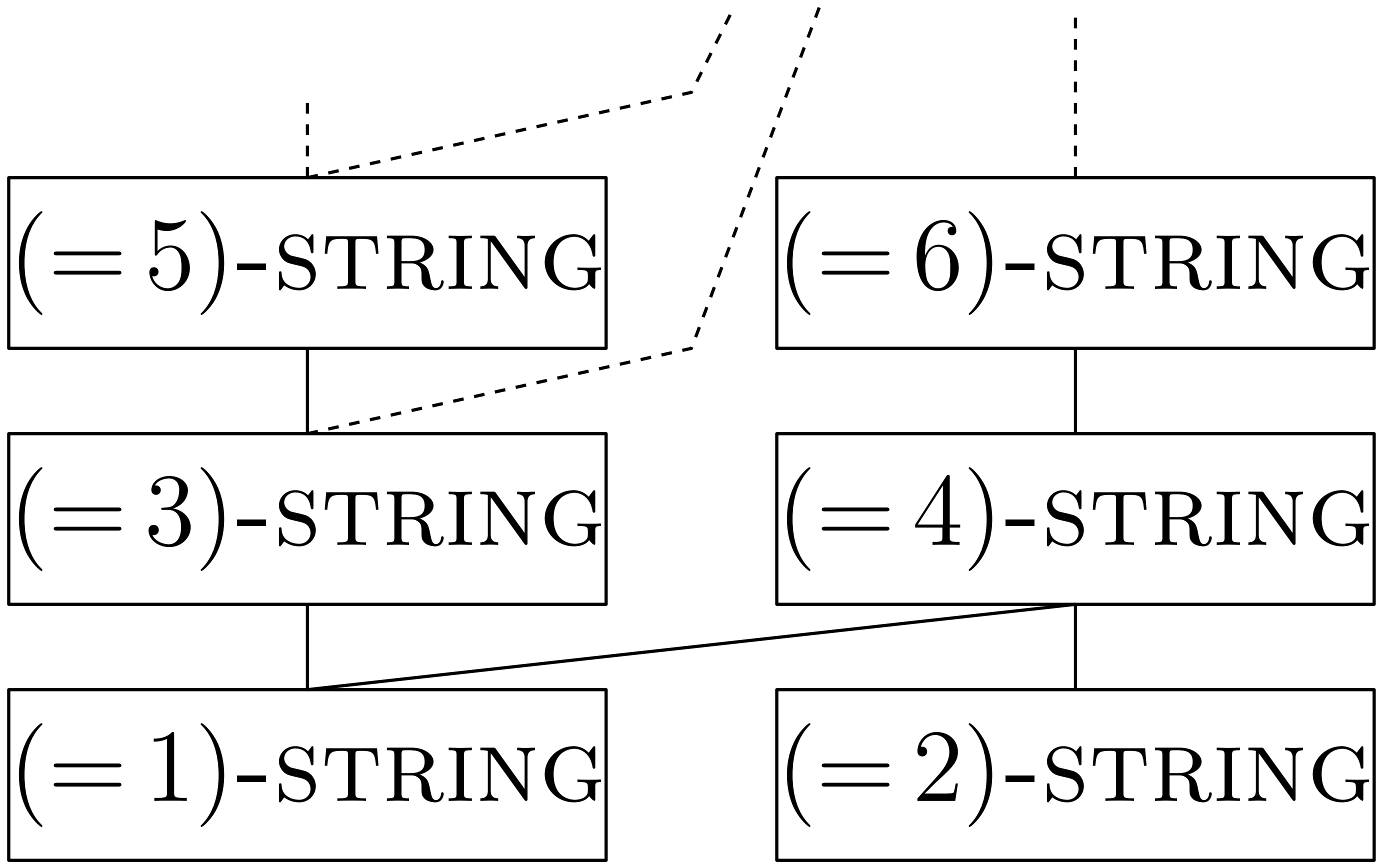

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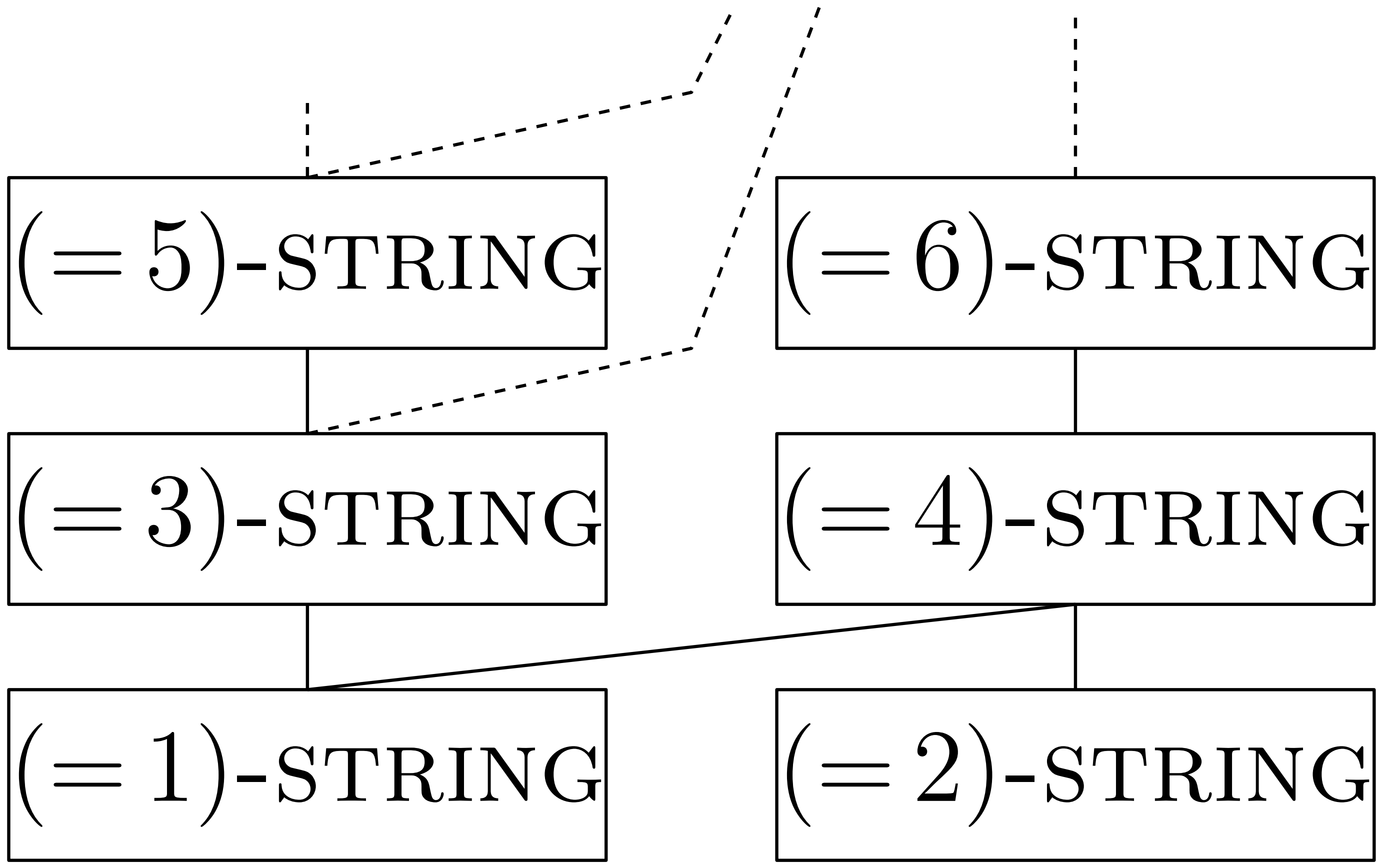

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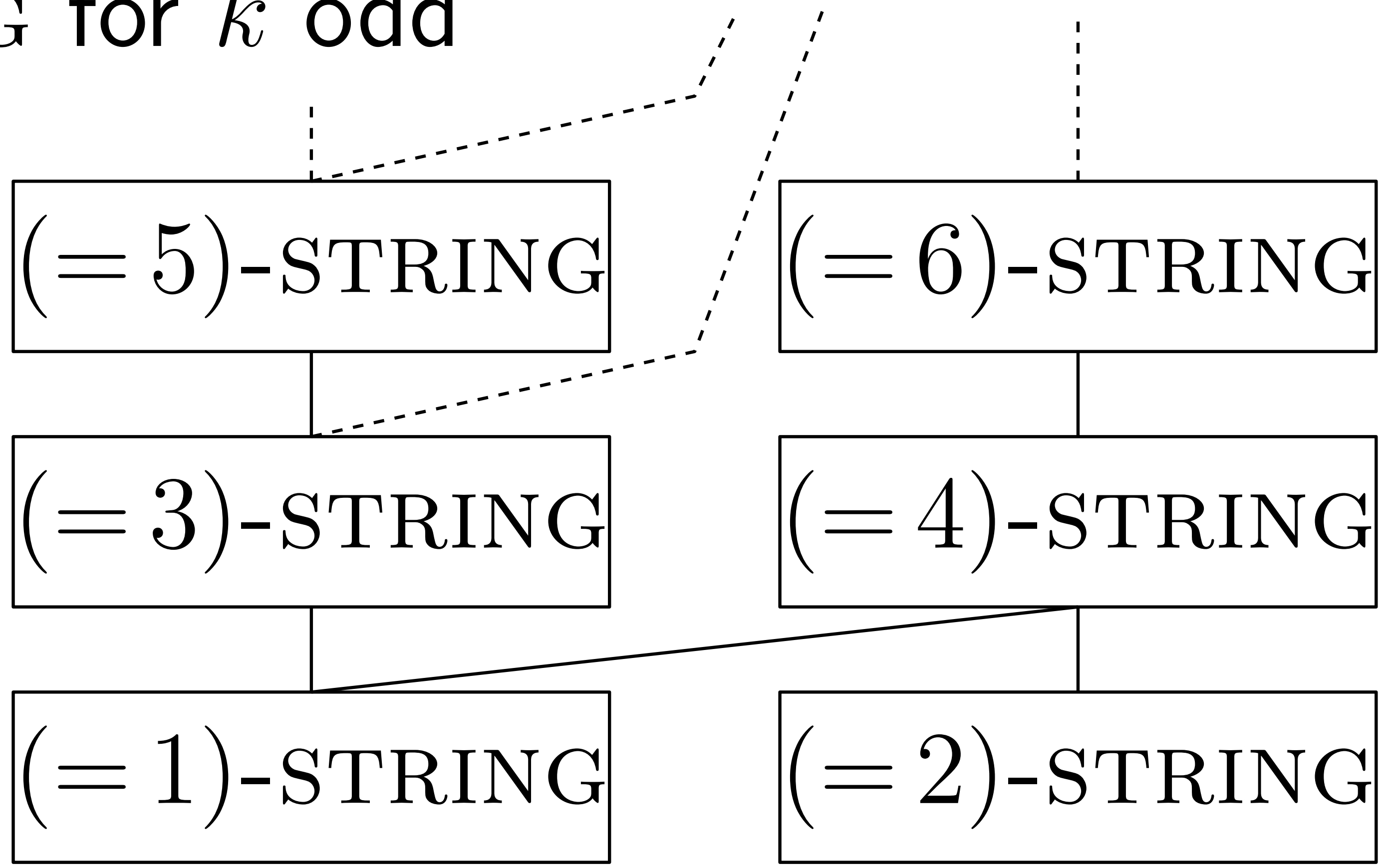

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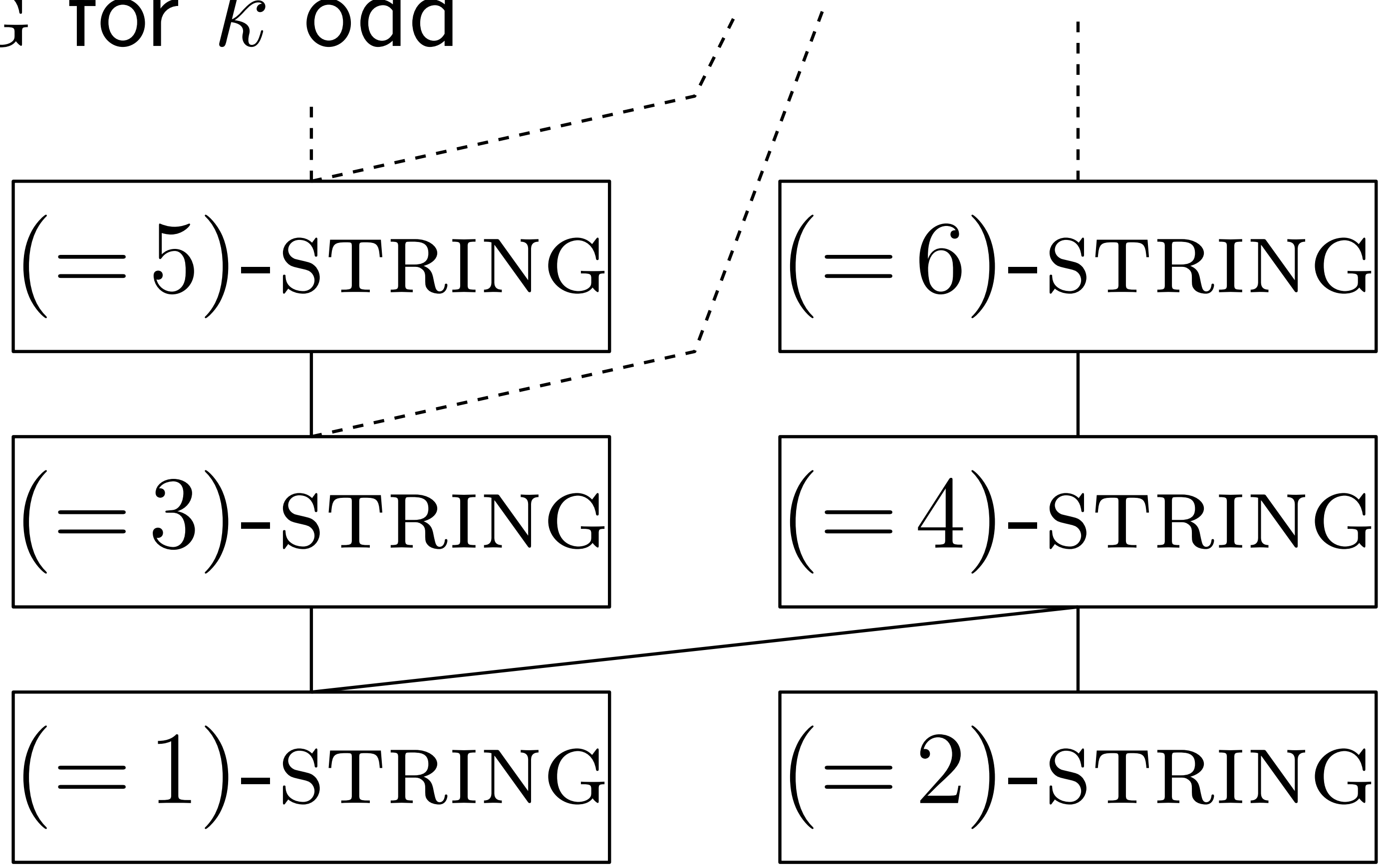
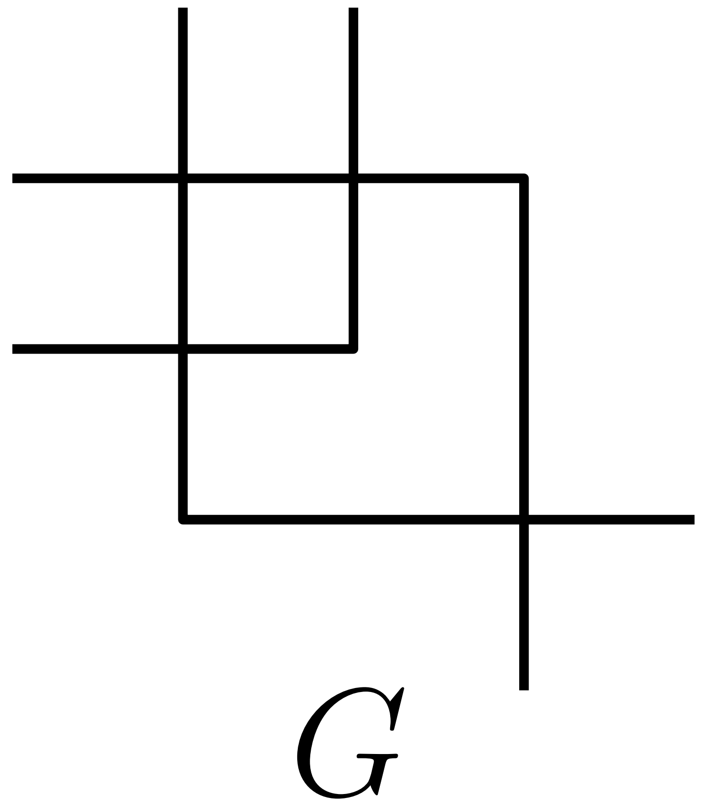


Tool: Noodle-Forcing Lemma

- Originally by Chaplick, Jelínek, Kratochvíl, Vyskočil ’12



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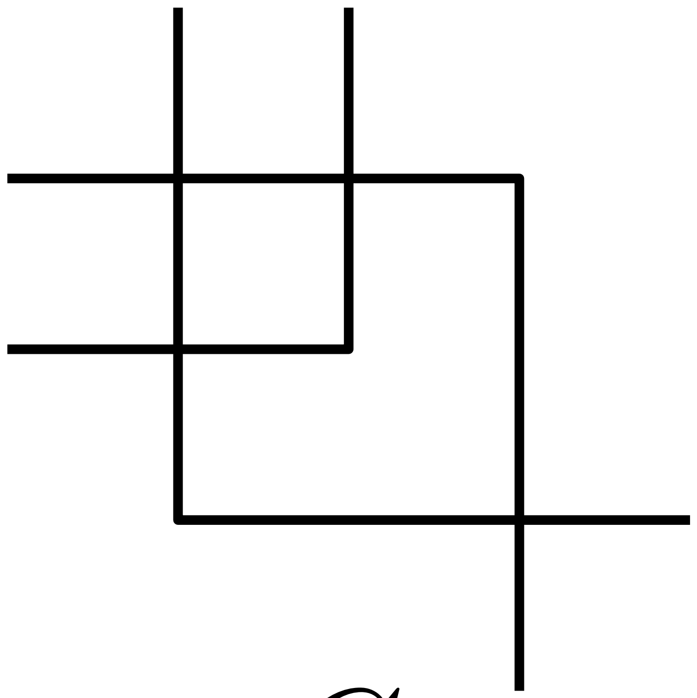

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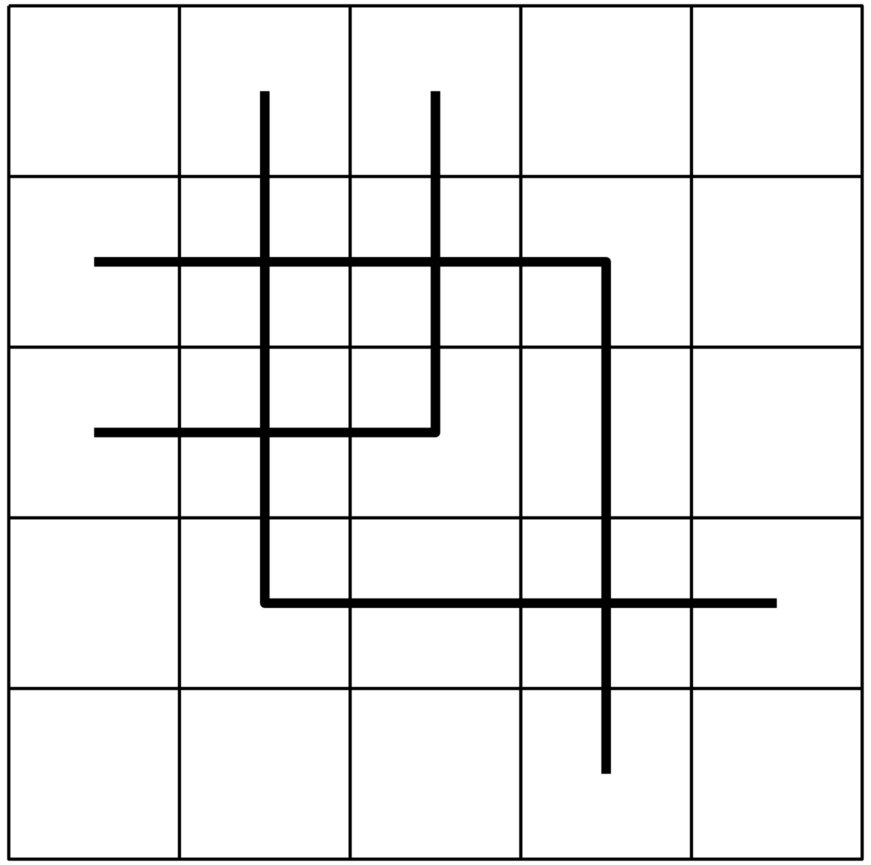

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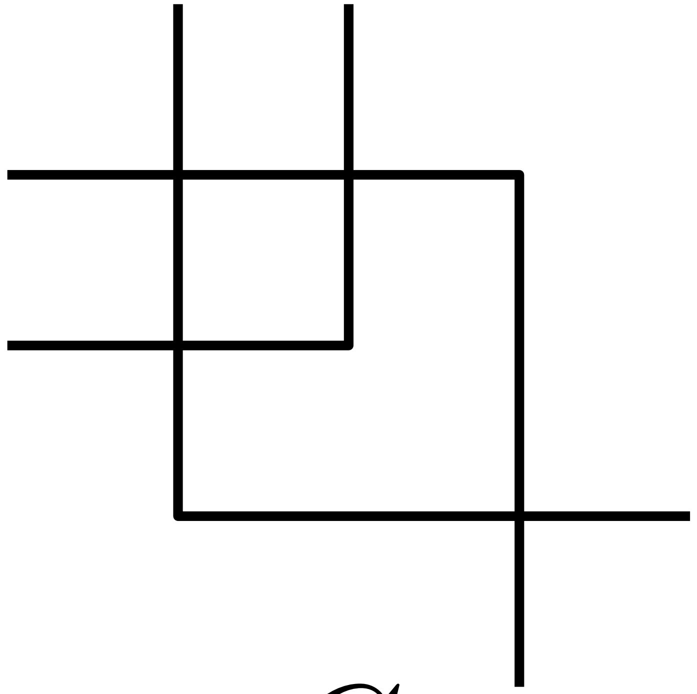

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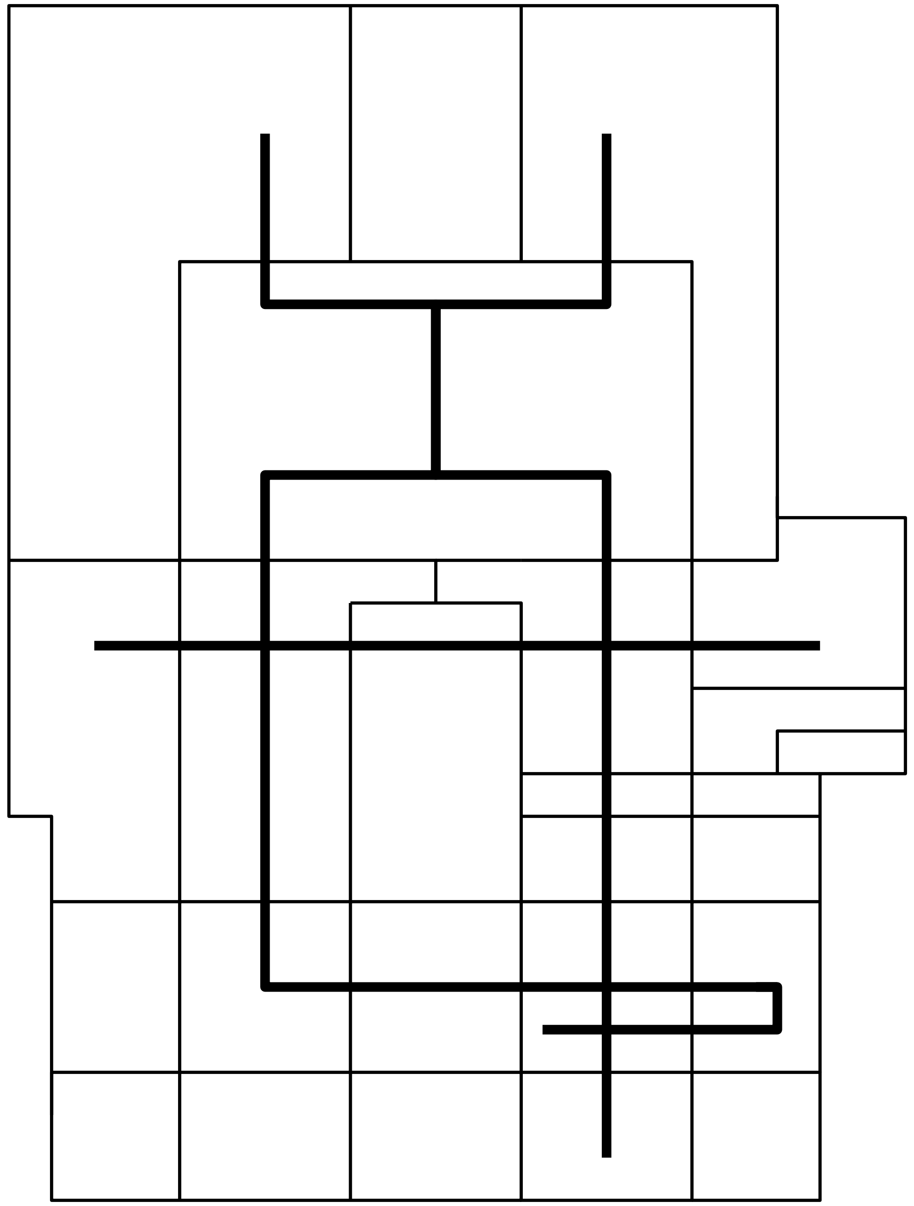

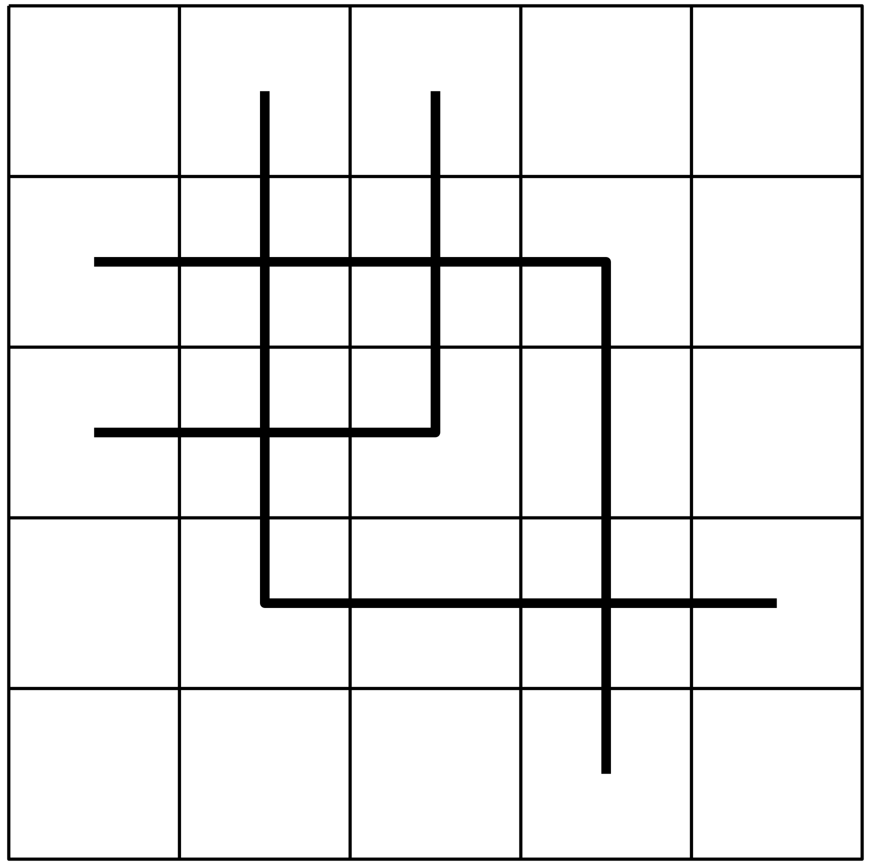

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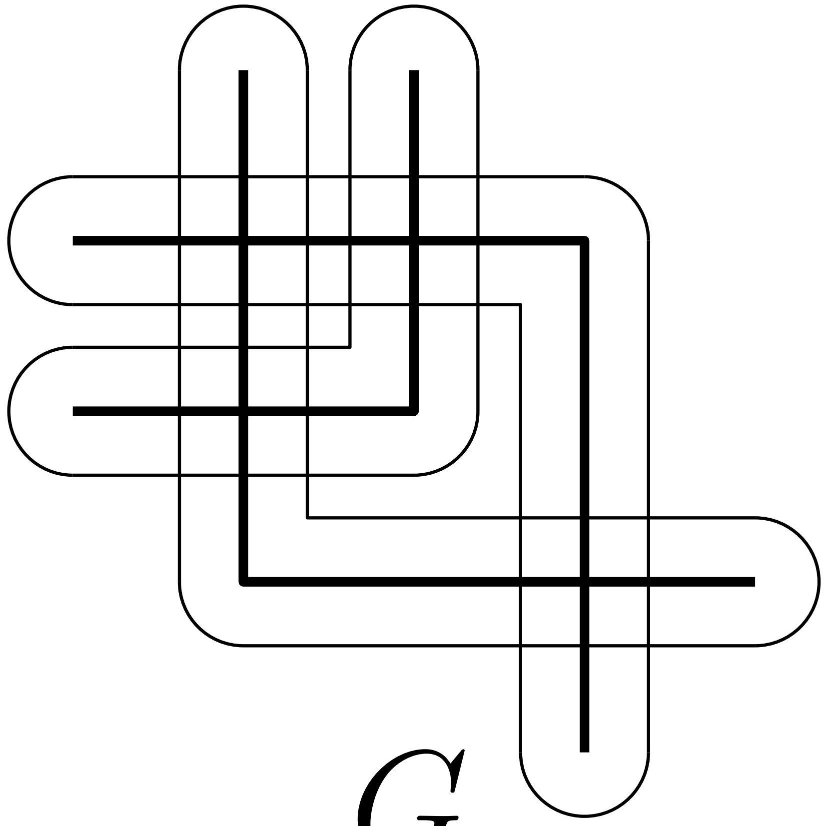

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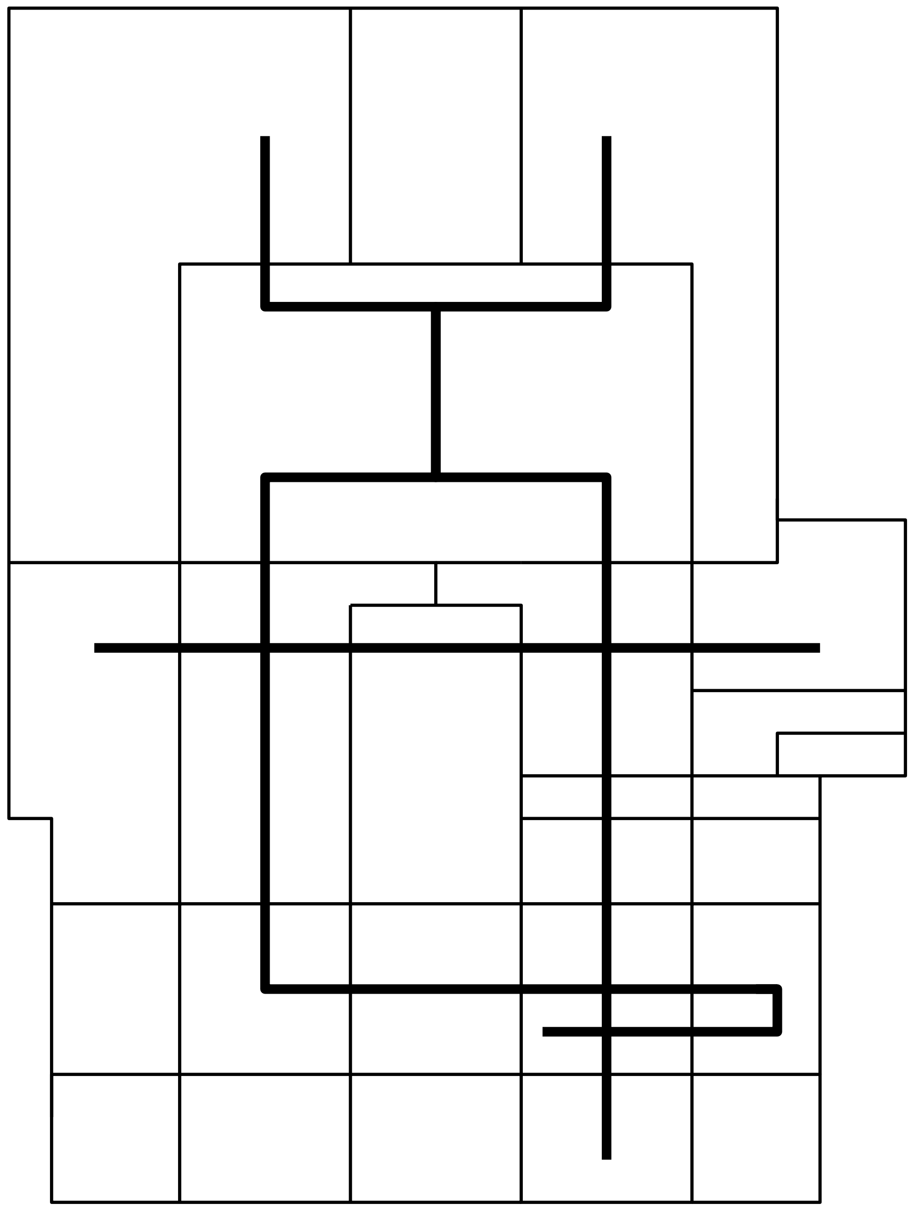

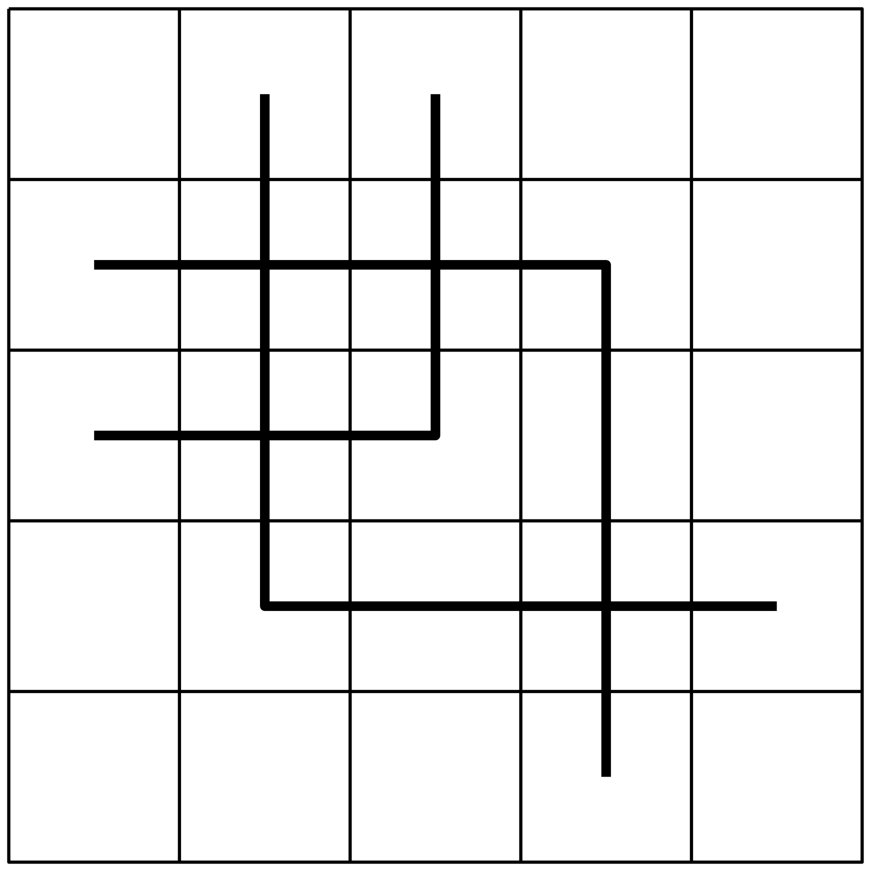

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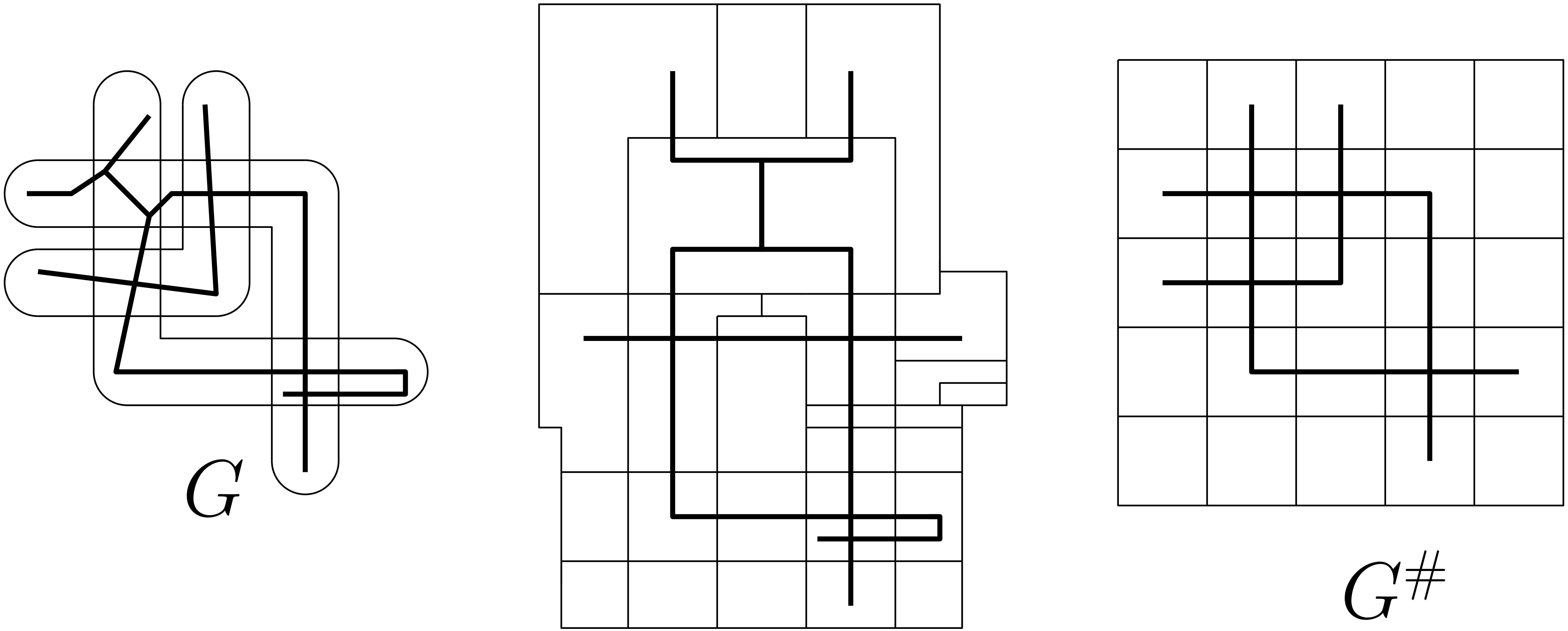

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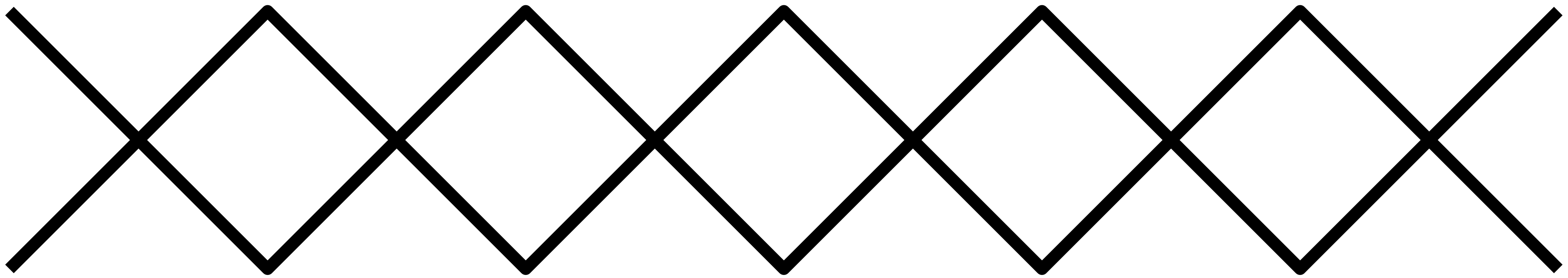

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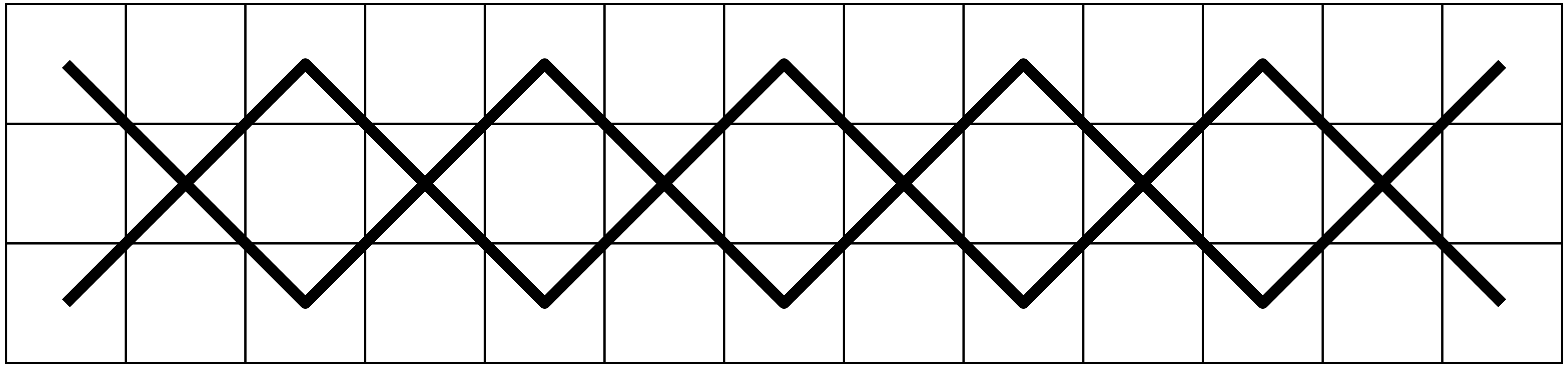

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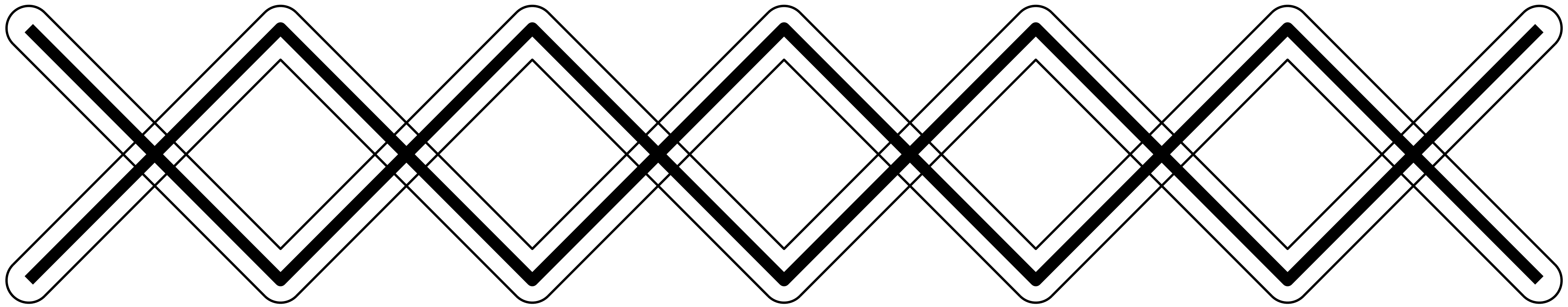

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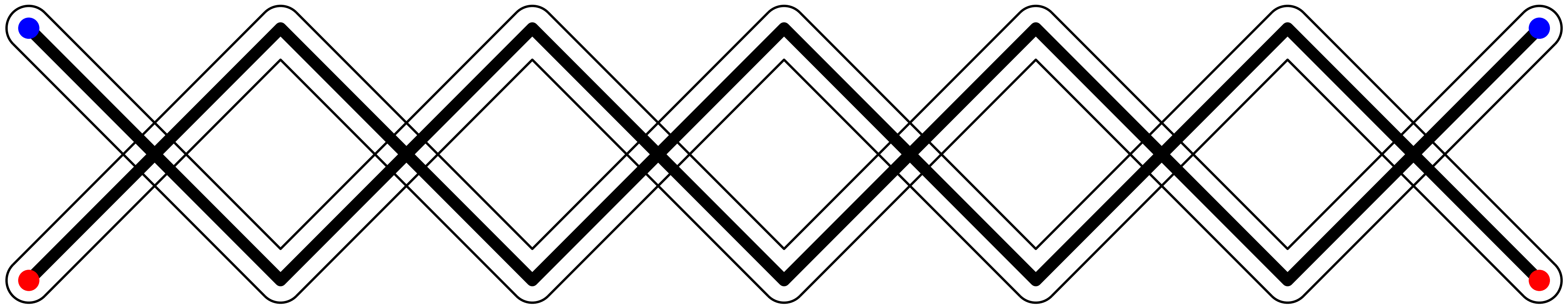

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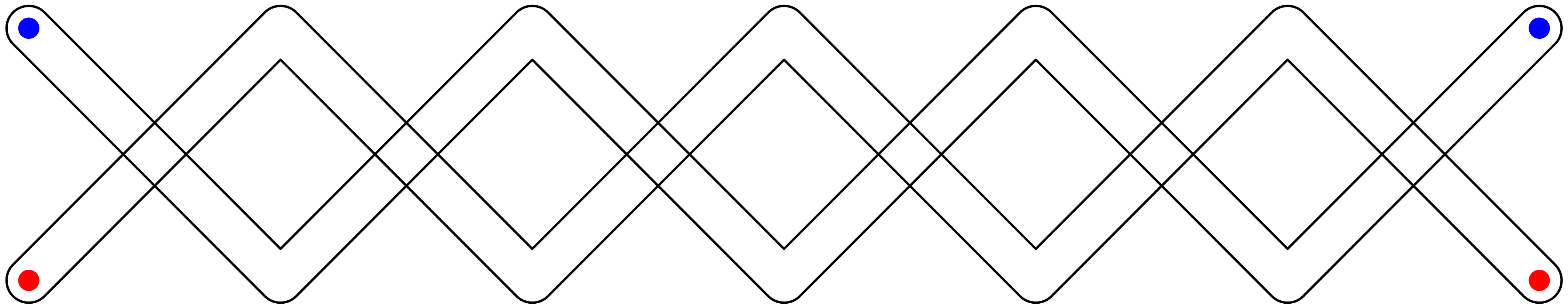

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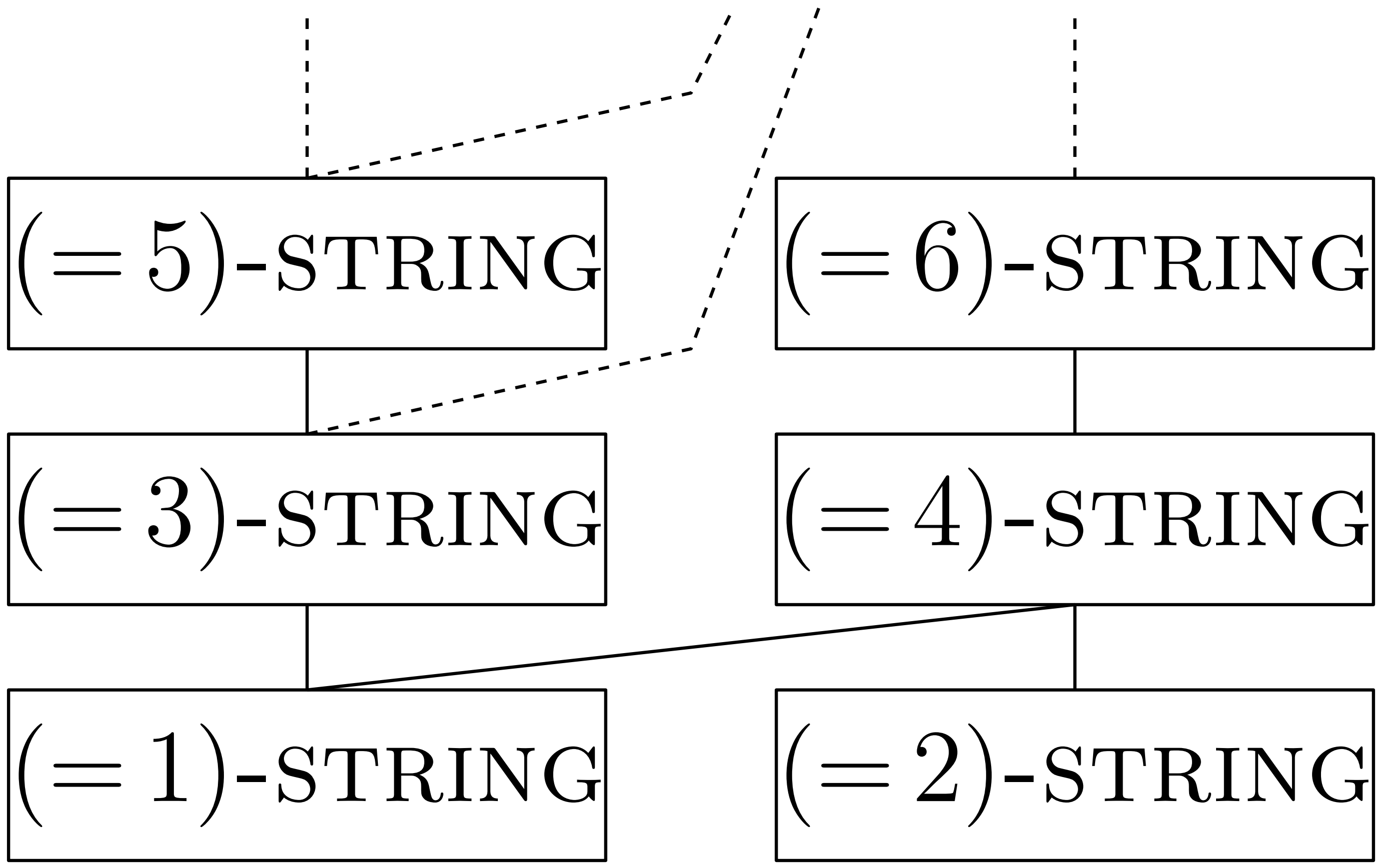

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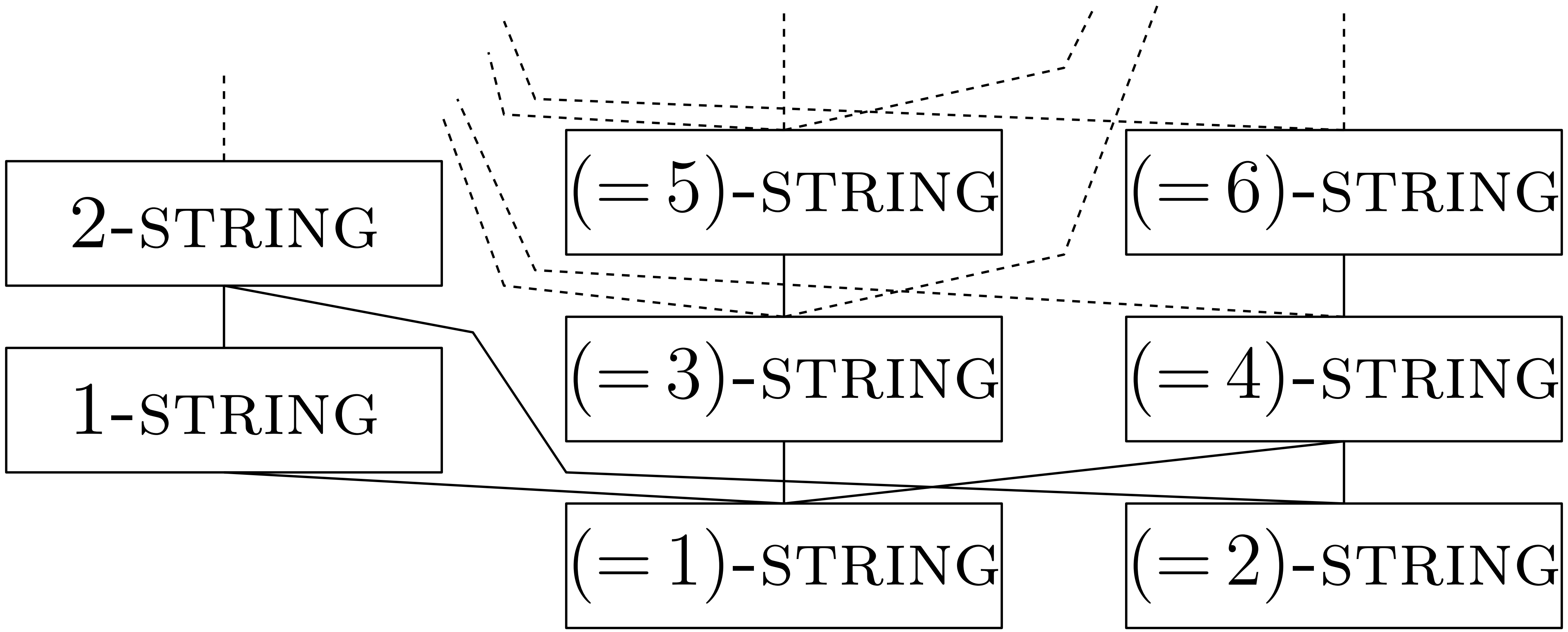

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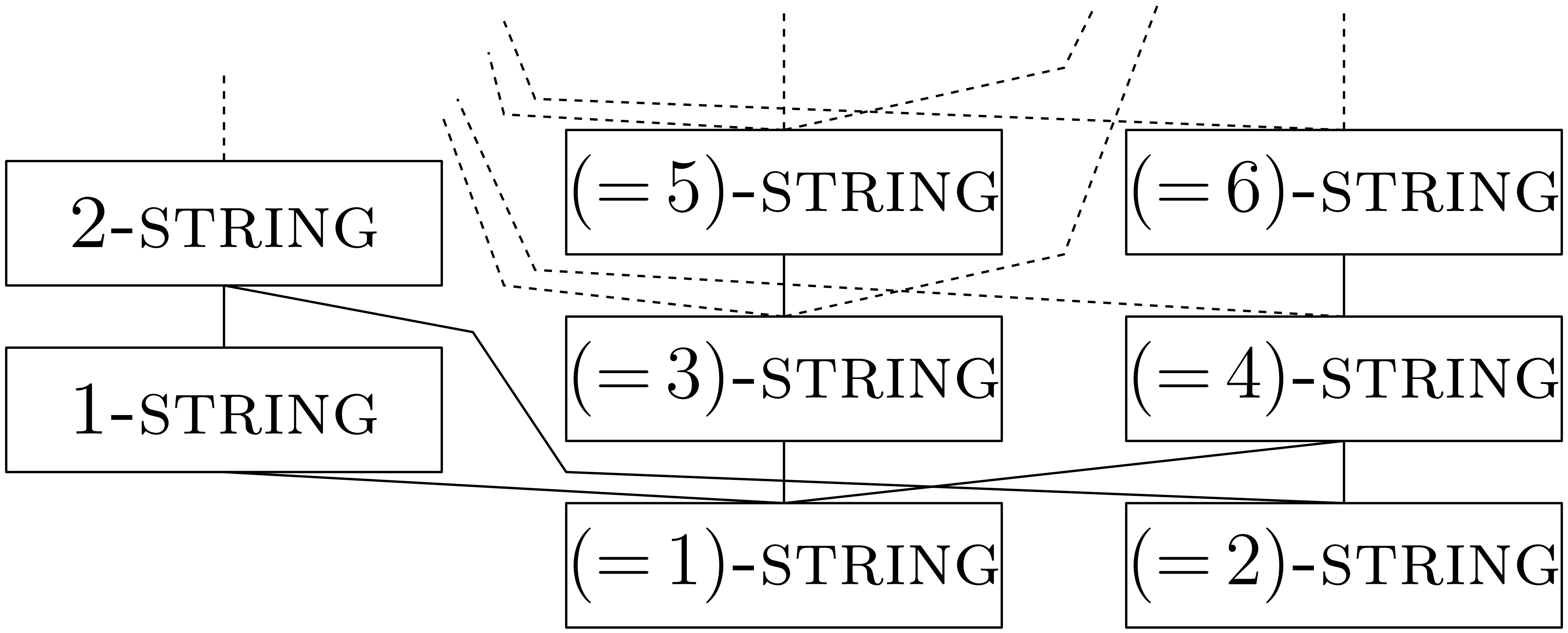

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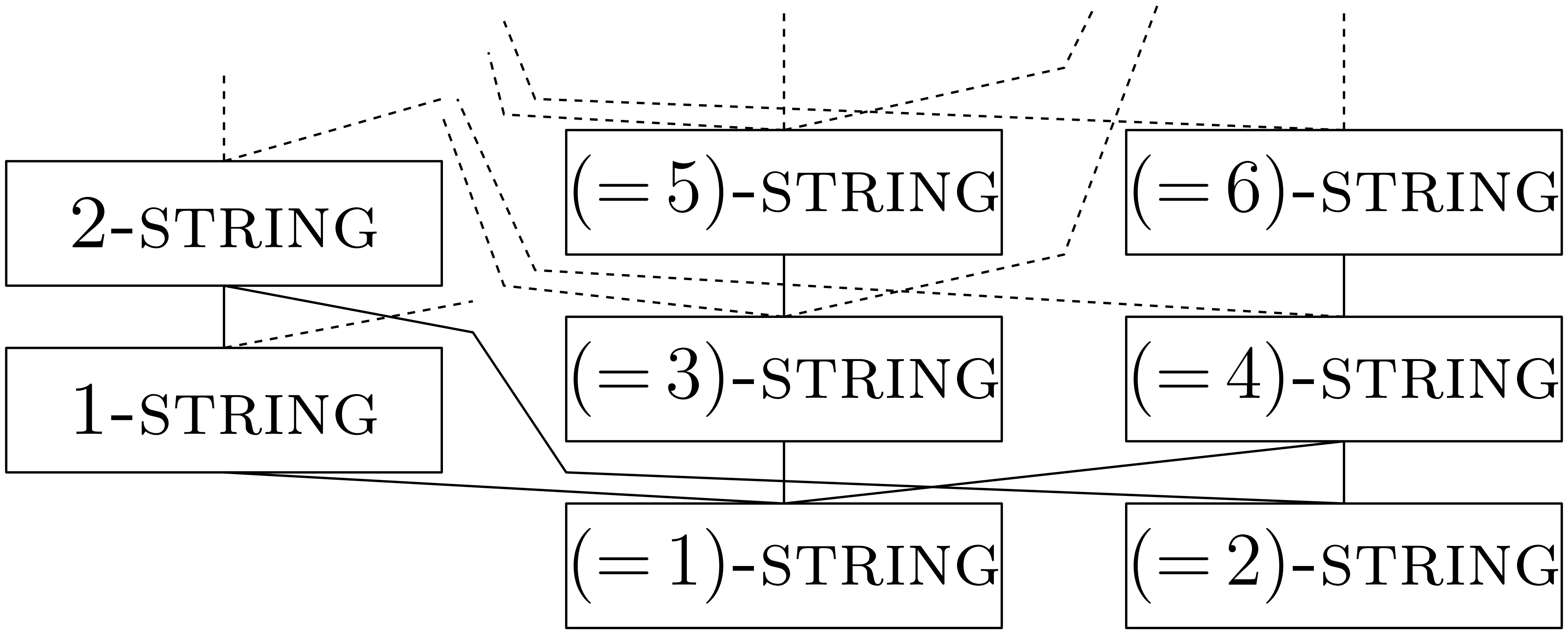

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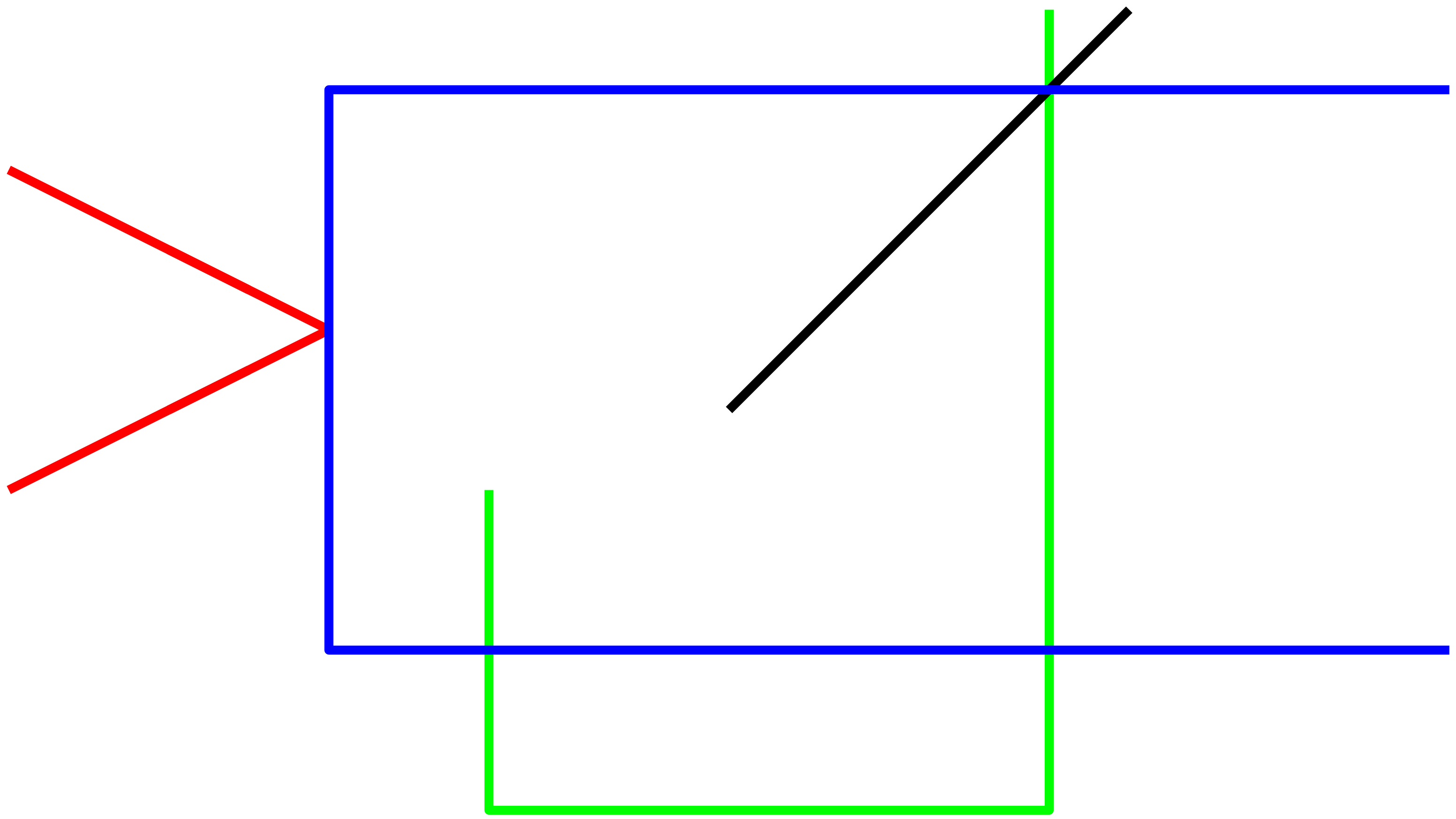

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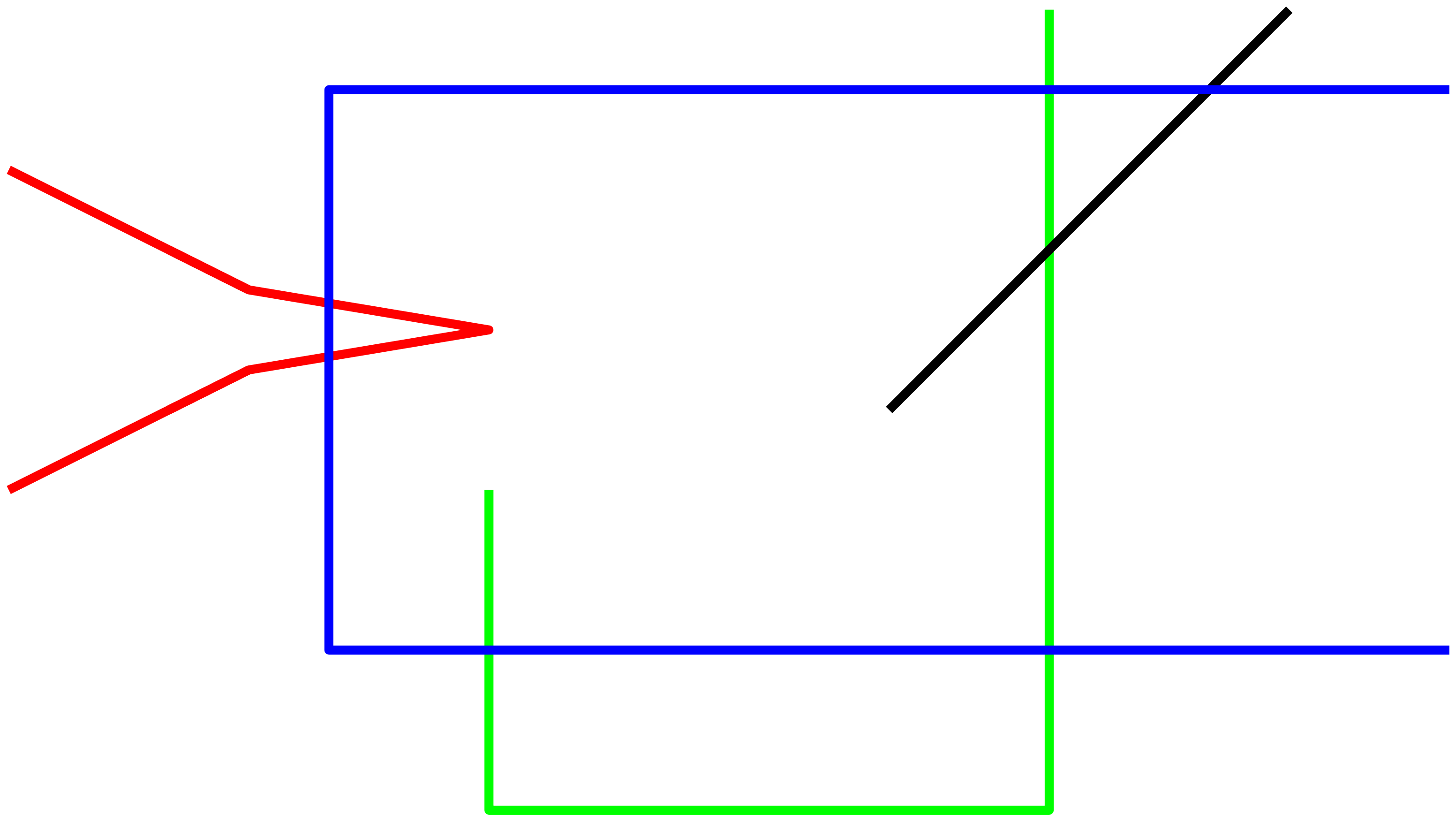

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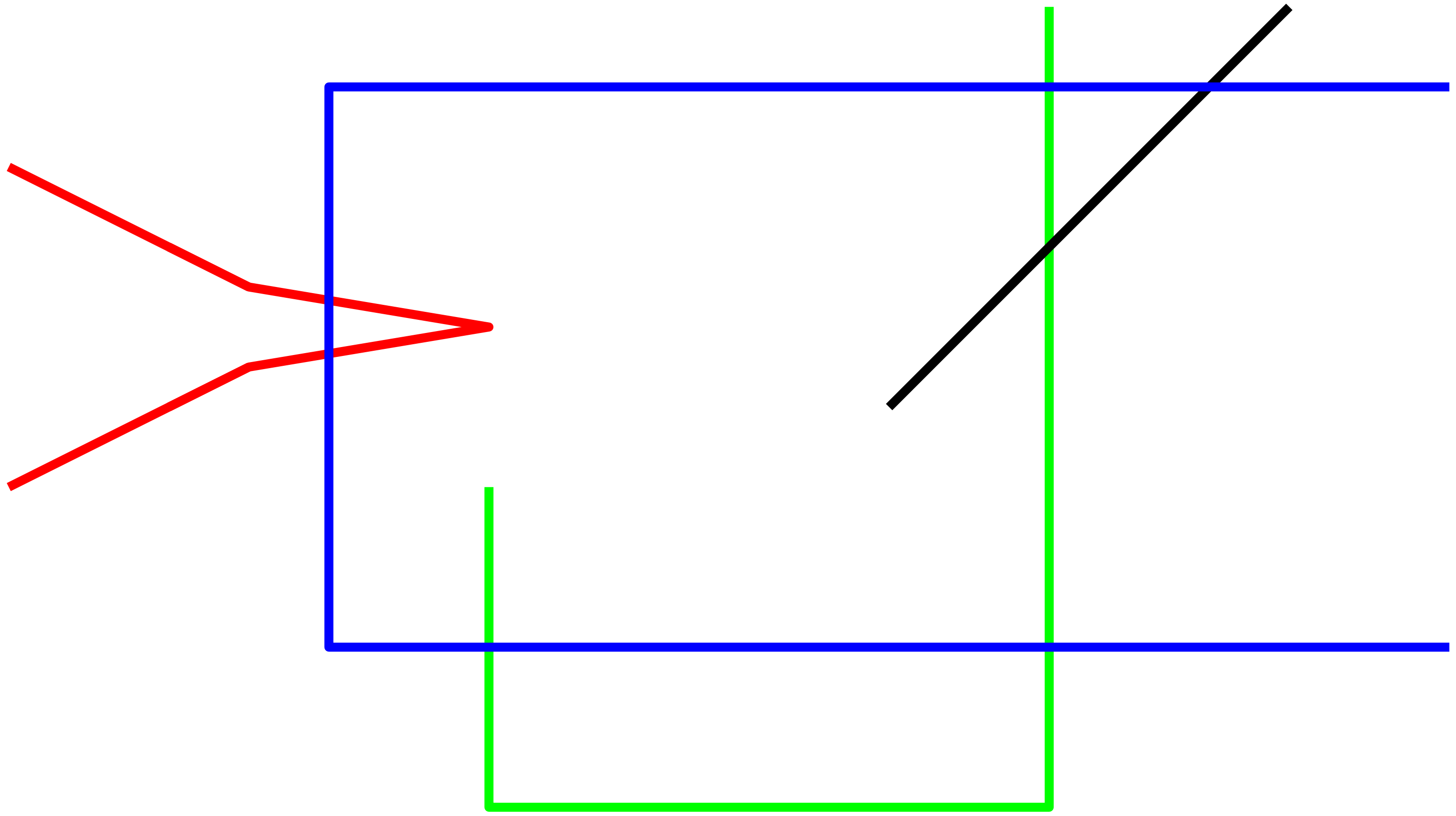

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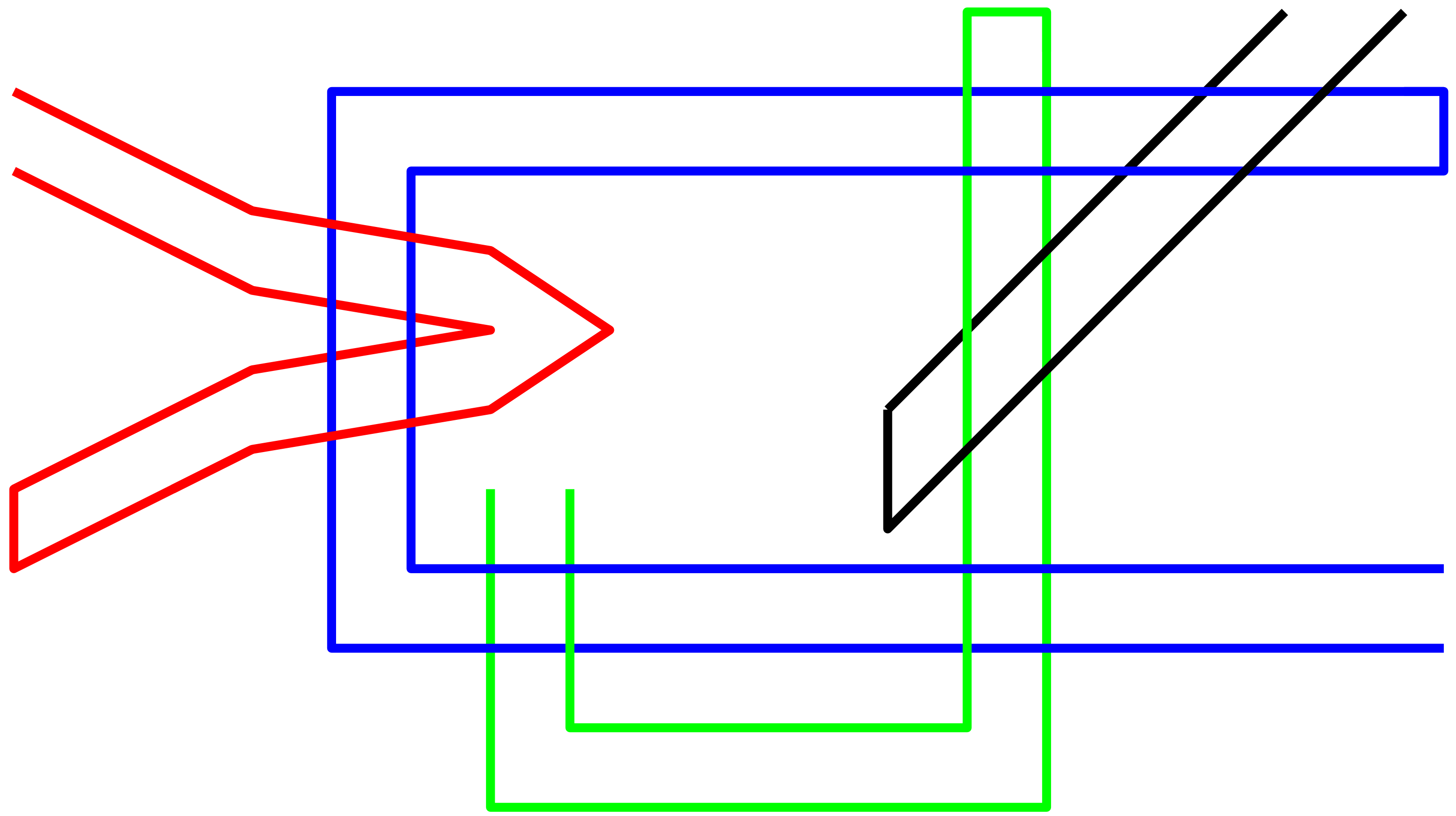

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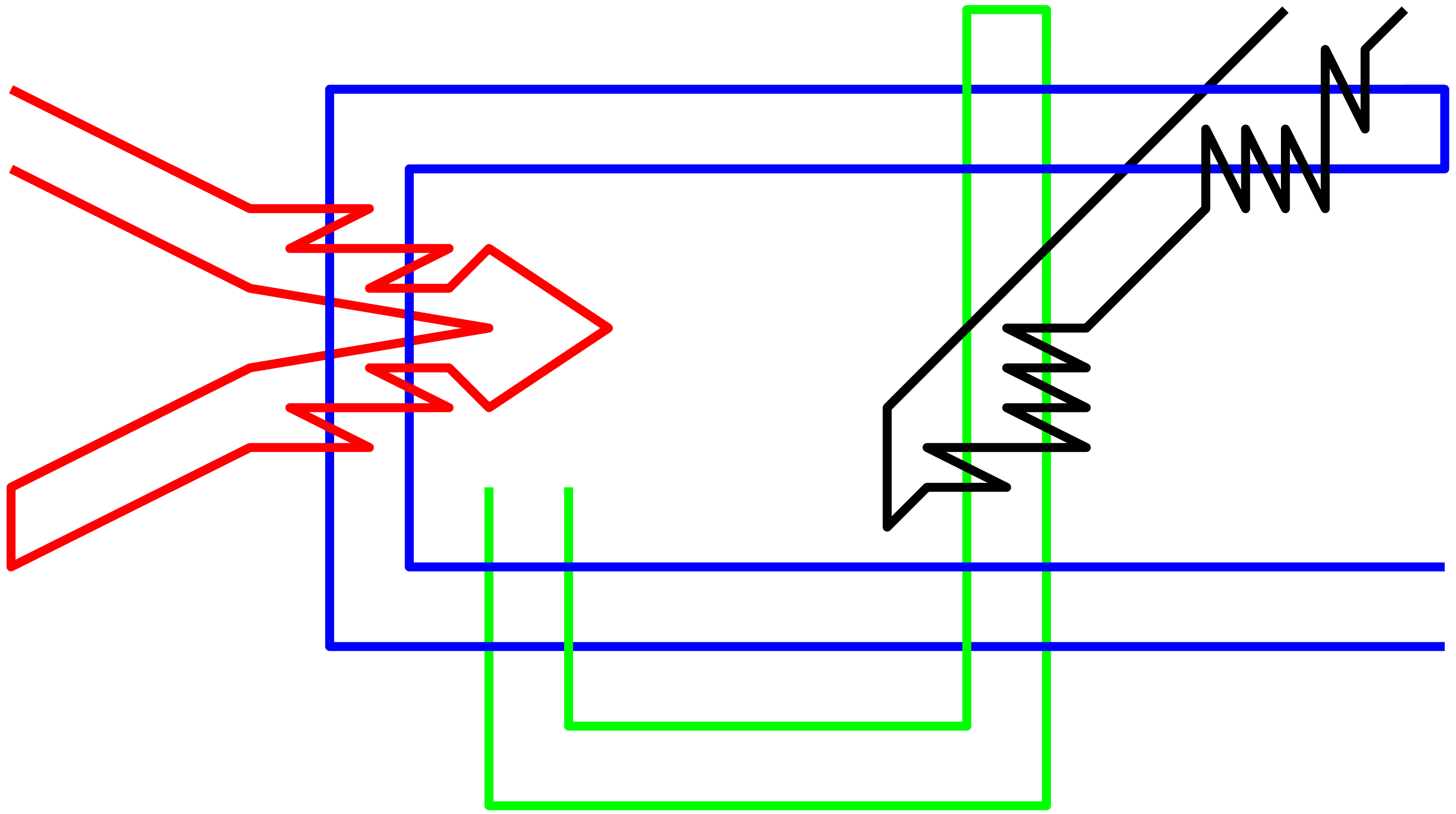

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Open question

Is this the best possible?

Odd-string graphs

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Open question

Is recognizing odd-string graphs in NP?
Weaker yet, is it decidable?

More open questions

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Thank you!