Degenerate crossing number and signed reversal distance

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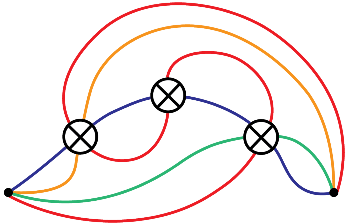

International Symposium on Graph Drawing
Palermo, September 2023

Outline

Crossing numbers for graphs

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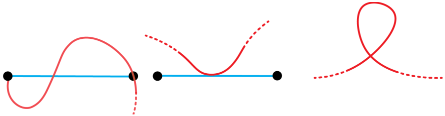

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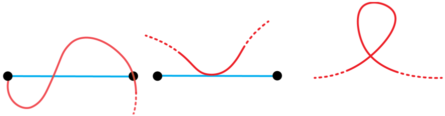

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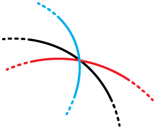

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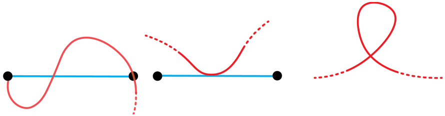

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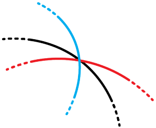

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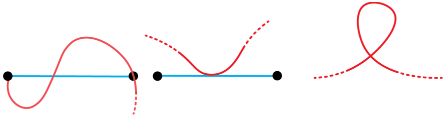

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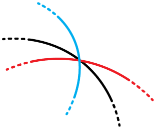

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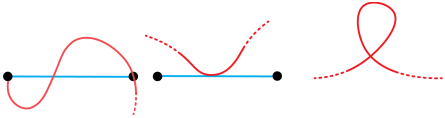

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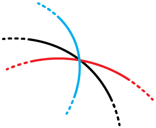

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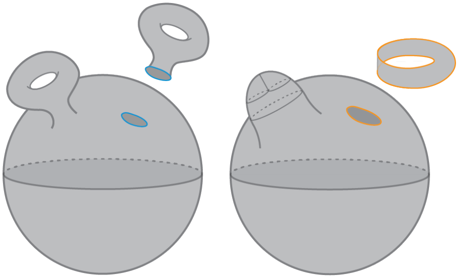
Mohar's Conjecture 1 ('07)

For every graph G, $\mathrm{gcr}(G) = \mathrm{dcr}(G)$.

Graphs embedded on surfaces

- A surface is a topological space that locally looks like the plane.
- In this talk, we deal with connected compact surfaces.
- They are classified by their orientability and their genus.



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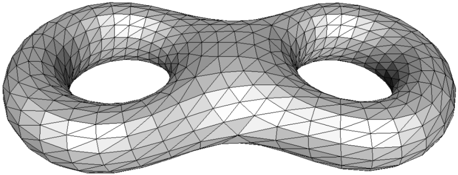

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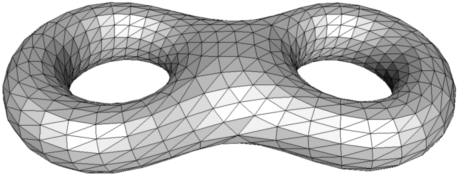

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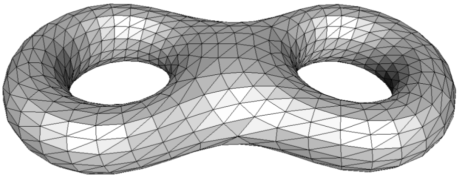
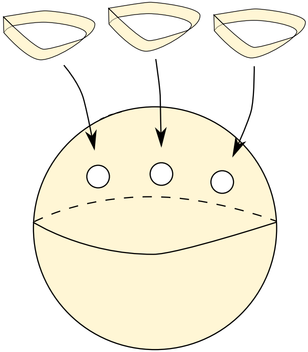


- The **non-orientable genus** $g(G)$ of a graph G is the minimum number of cross-caps that it needs to be embedded on a surface.
- Graph embeddings are hard to visualize on a surface.

Cross-cap drawings and non-orientable embeddings

- One can represent a non-orientable embedding by a planar drawing.



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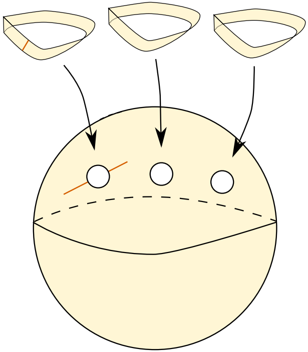

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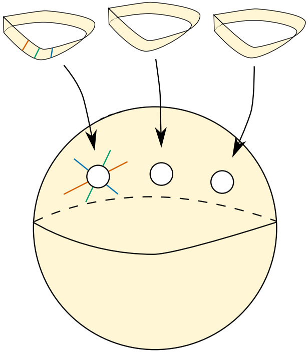

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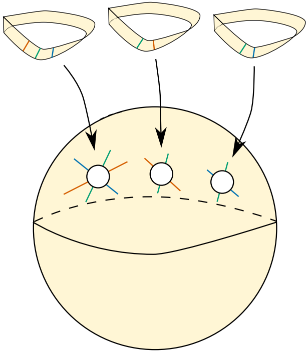

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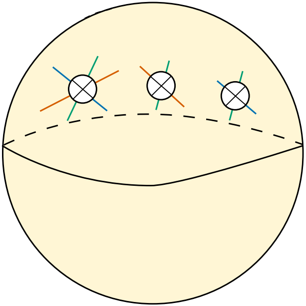

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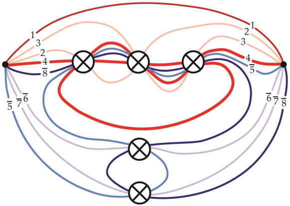

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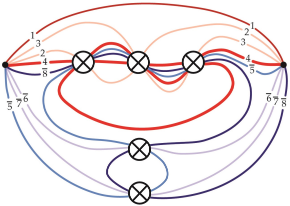

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Question

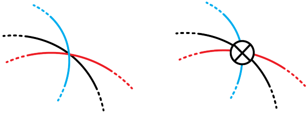
Can we control the number of times an edge enters a cross-cap?

From crossing numbers to non-orientable genus

These cross-caps can be interpreted as multiple transverse crossings.

Theorem (Mohar '07)

For any graph G, $gcr(G) =$ non-orientable genus of G.



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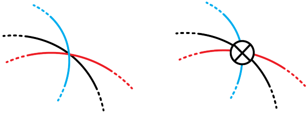

A **perfect cross-cap drawing** for a graph is one in which each edge enters each cross-cap **at most once**.

Mohar's Conjecture 1 ('07)

For every graph G, $dcr(G) = gcr(G) = g(G)$.

$\Downarrow$

Every graph G admits a **perfect** cross-cap drawing with $g(G)$ cross-caps.

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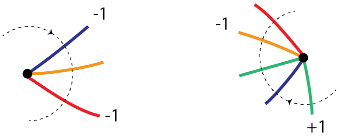
Theorem (F.,Hubard,de Mesmay '23)

Apart from two exceptional families of graphs, all the 2-vertex loopless graphs embedded on non-orientable surfaces satisfy Conjecture 2.

Embedding schemes

- An embedding for a graph, is entirely described by **an embedding scheme**:
    - the cyclic ordering of the edges around the vertex
    - (in the non-orientable case) a signature $+1$ or -1 associated to each edge



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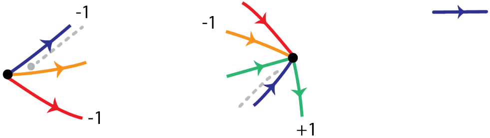

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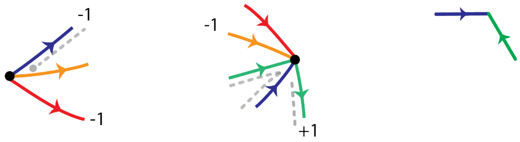

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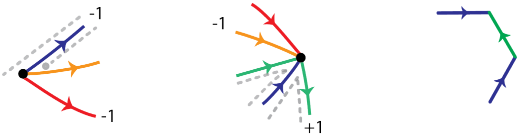

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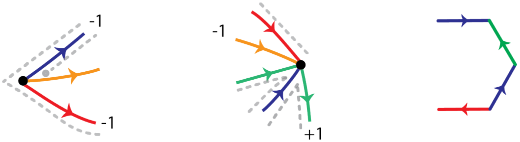

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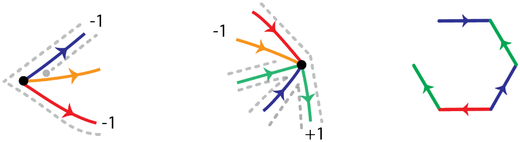

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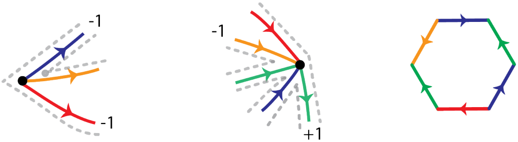

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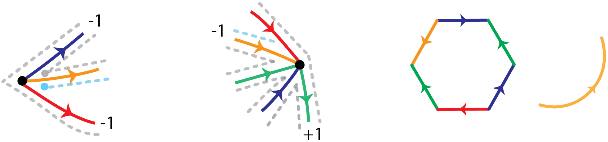

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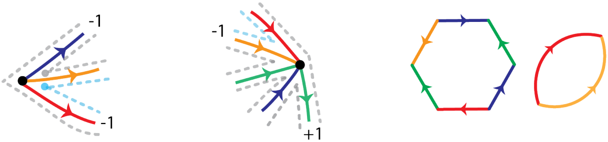

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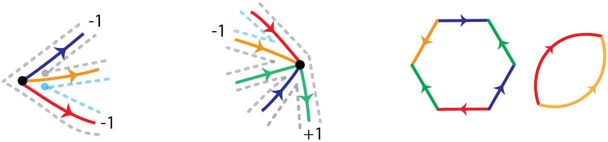
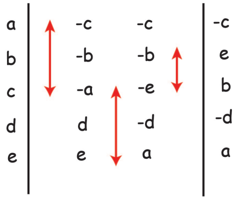


- A cross-cap drawing of an embedding scheme respects the signatures: each edge with signature $+1$ (resp. -1) enters even (resp. odd) number of cross-caps.

An unexpected connection

Our main technical tool for our results comes from computational biology.

- The signed reversal distance between two signed words is the minimum number of reversals to go from one to the other one.
- Very important in computational genomics, computable in polynomial time [Hannenhalli-Pevzner '99].
- Strong similarities with crosscap drawings, which we leverage in all of our results.



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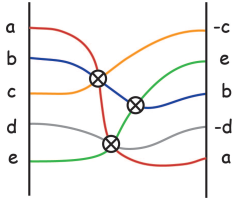

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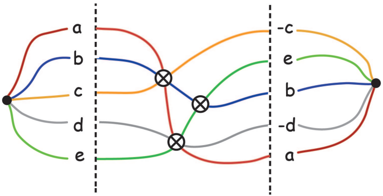

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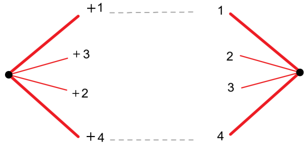
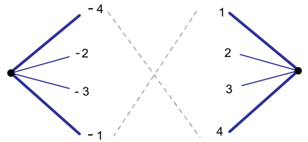
→ dealing with these sub-words costs them extra cross-caps:

- **Positive block**:
  - The **frames** 1 and 4 appear with 14 and 41 order around vertices.
  - all $+1$ signatures.



- **Negative block**:
  - The **frames** 1 and 4 appear with 14 around both vertices.
  - all -1 signatures.



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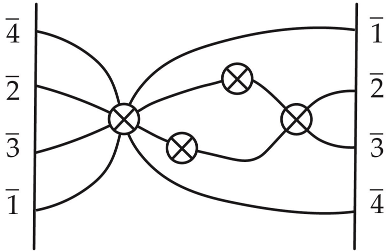

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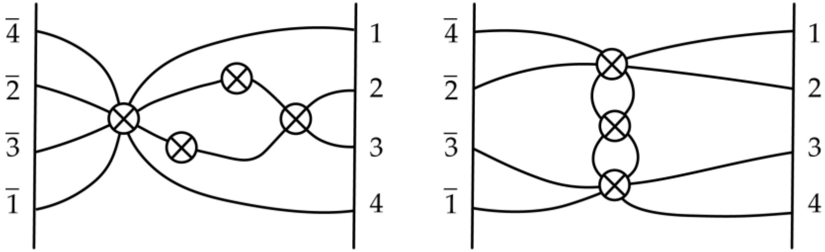


The counter example

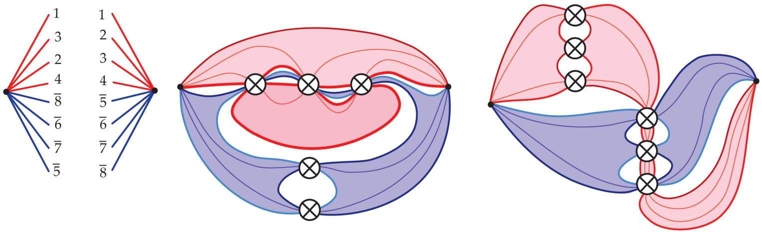
Mohar's (stronger) Conjecture 2 ('07)

Every loopless graph embedded on a non-orientable surface admits a **perfect** cross-cap drawing.

Conjecture 2 does not hold:

Theorem (F., Hubard, de Mesmay '23)

*There exists a 2-vertex loopless graph embedded on a non-orientable surface that does not admit a **perfect** cross-cap drawing.*



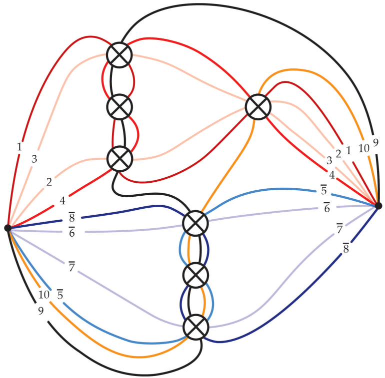
Cross-cap drawings of 2-vertex schemes

Theorem (F., Hubard, de Mesmay '23)

Apart from two exceptional families of graphs, all the 2-vertex loopless graphs embedded on non-orientable surfaces satisfy Conjecture 2.

In particular:

- Under standard models of random maps, almost all 2-vertex loopless embedded graphs satisfy Conjecture 2.
- The behavior under adding edges is counter-intuitive.



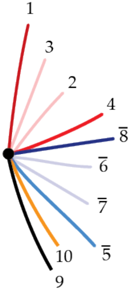
Cross-cap drawings of 2-vertex schemes

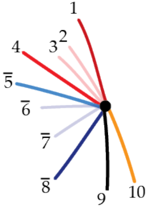
Theorem (F., Hubard, de Mesmay '23)

Apart from two exceptional families of graphs, all the 2-vertex loopless graphs embedded on non-orientable surfaces satisfy Conjecture 2.

Sketch of the proof:

→ **reduce** the scheme.





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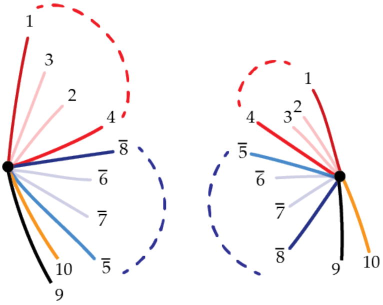

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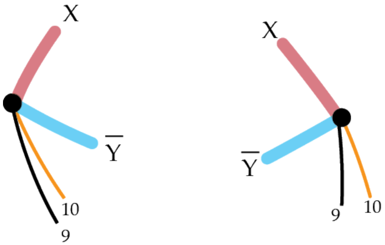

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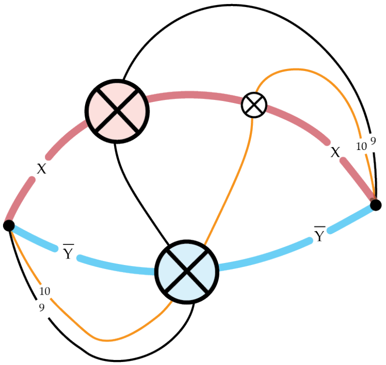

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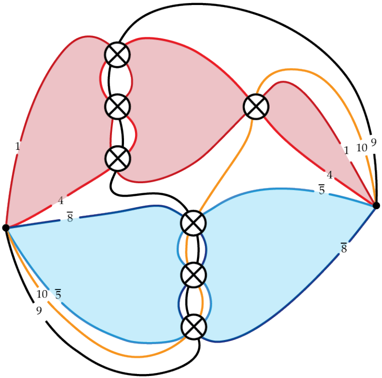
→ **blow up** the cross-caps.



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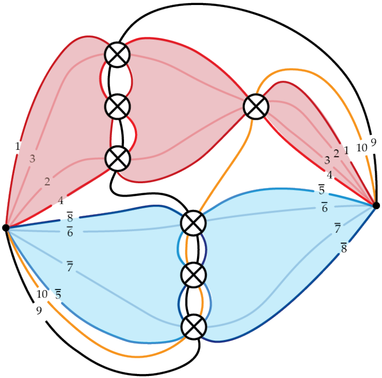

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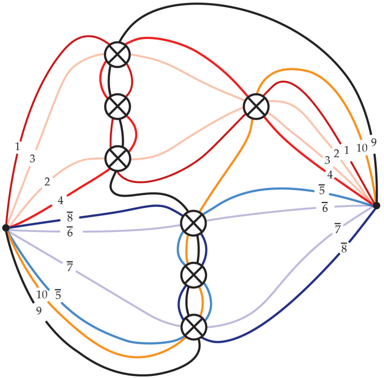
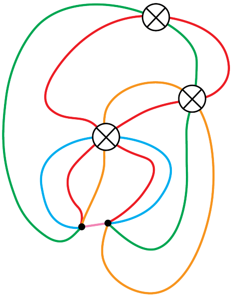


Conclusion

- Allowing the graph to have more vertices, increases the possibility of having a **perfect** cross-cap drawing.



→ Although Mohar's conjectures 2 and 3 are wrong, there is a great chance that conjecture 1 is correct.

Mohar's Conjecture 1 ('07)

For every graph G, $\text{gcr}(G) = \text{dcr}(G)$.

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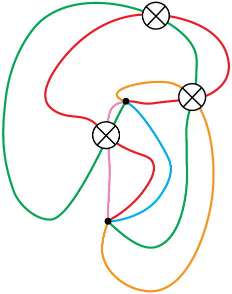

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Thank You!