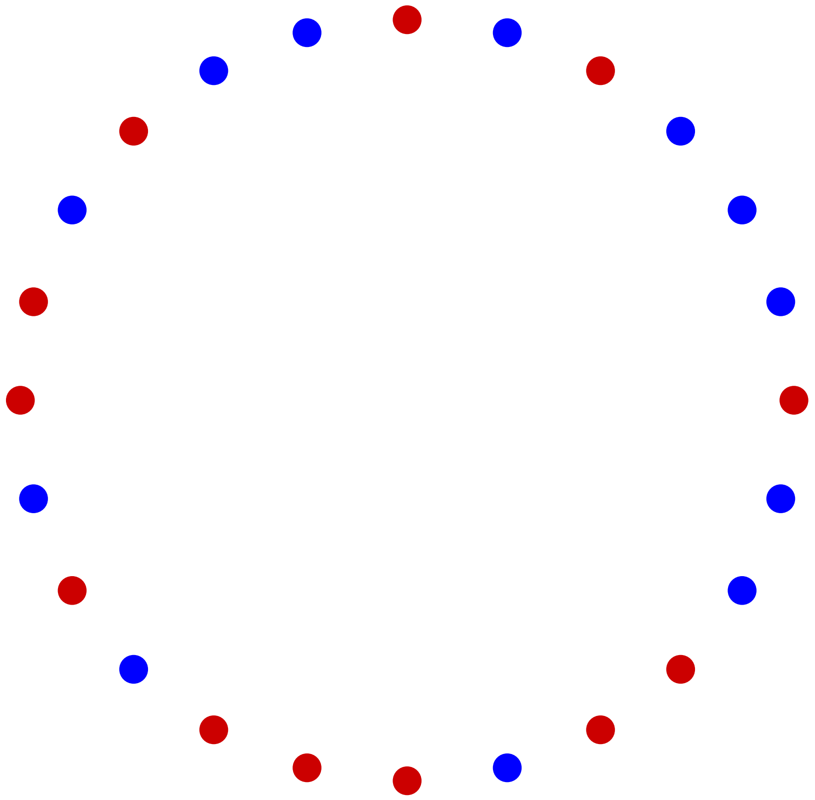
Bichromatic Perfect Matchings with Crossings

Oswin Aichholzer, Stefan Felsner, Rosna Paul,
Manfred Scheucher, Birgit Vogtenhuber

Setting

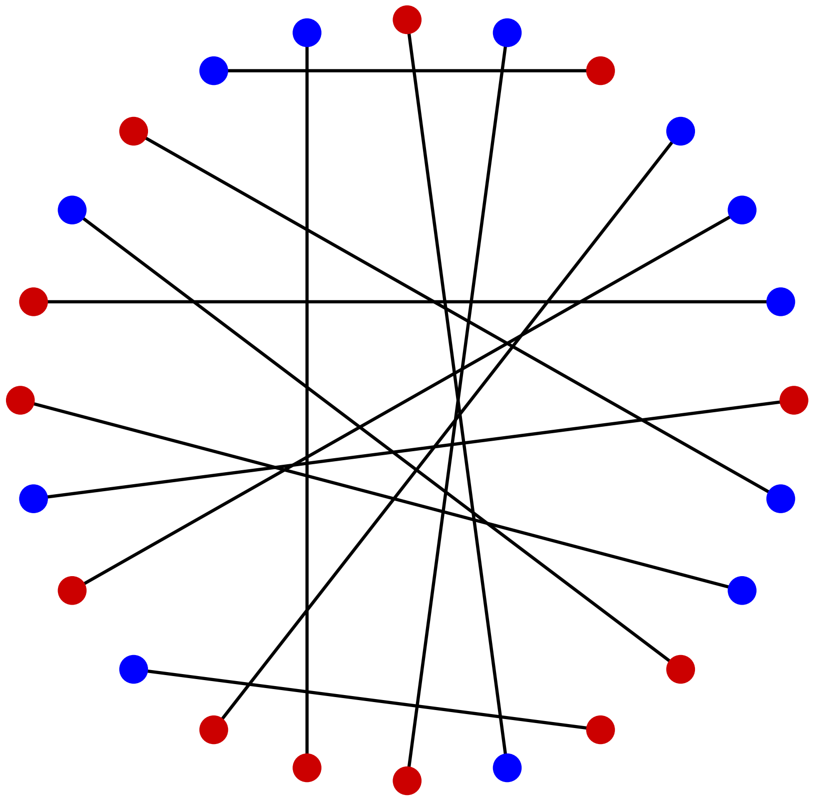
Setting

Points in convex position ($P = R \cup B$, $|R| = |B| = n$)



Setting

Straight-line Bichromatic Perfect Matchings with crossings



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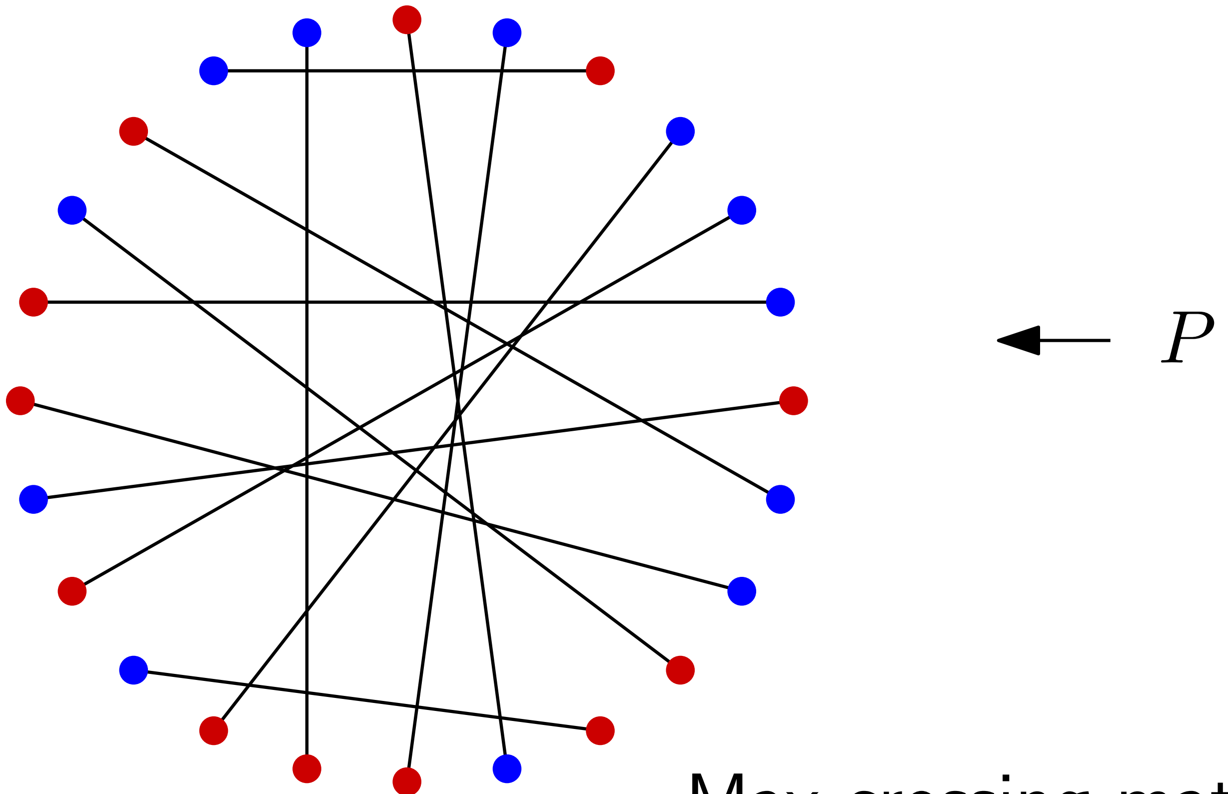

Max-crossing matching, $M_P^{\vee}$

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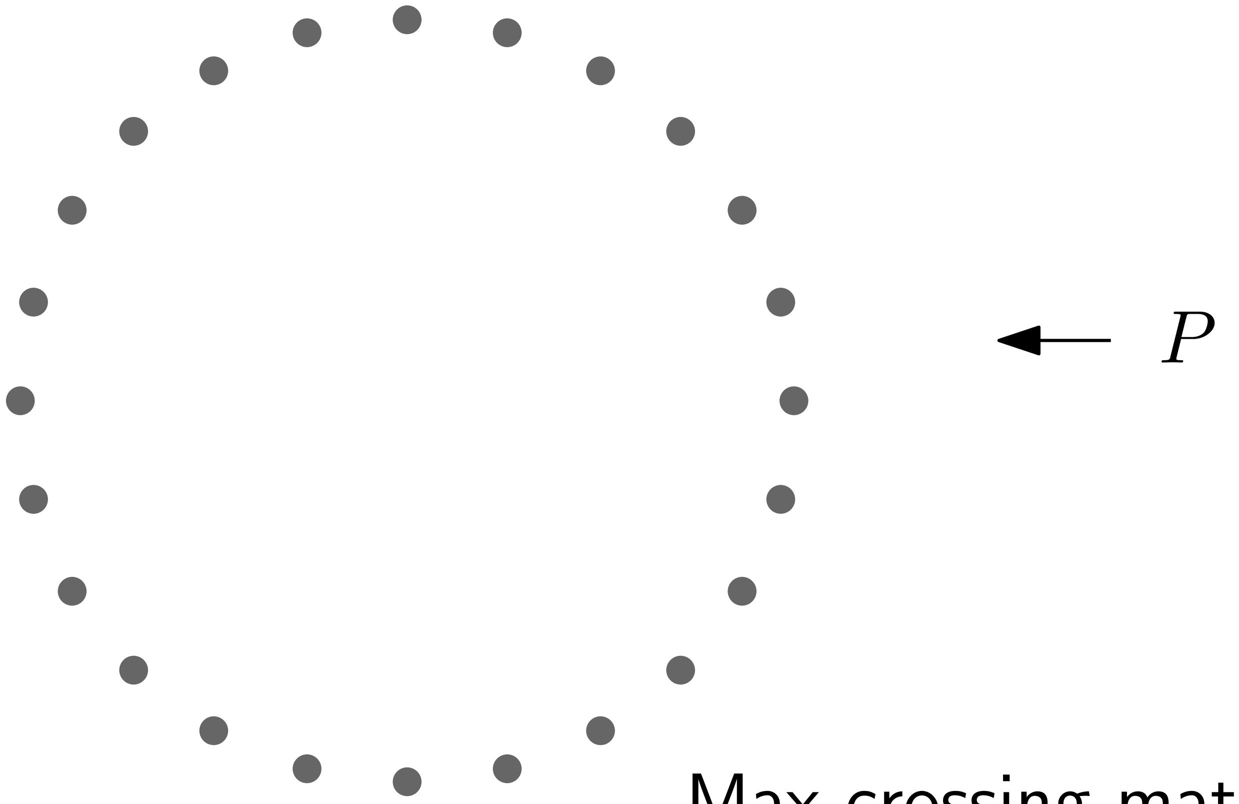

Max-crossing matching, $M_P^{\vee}$

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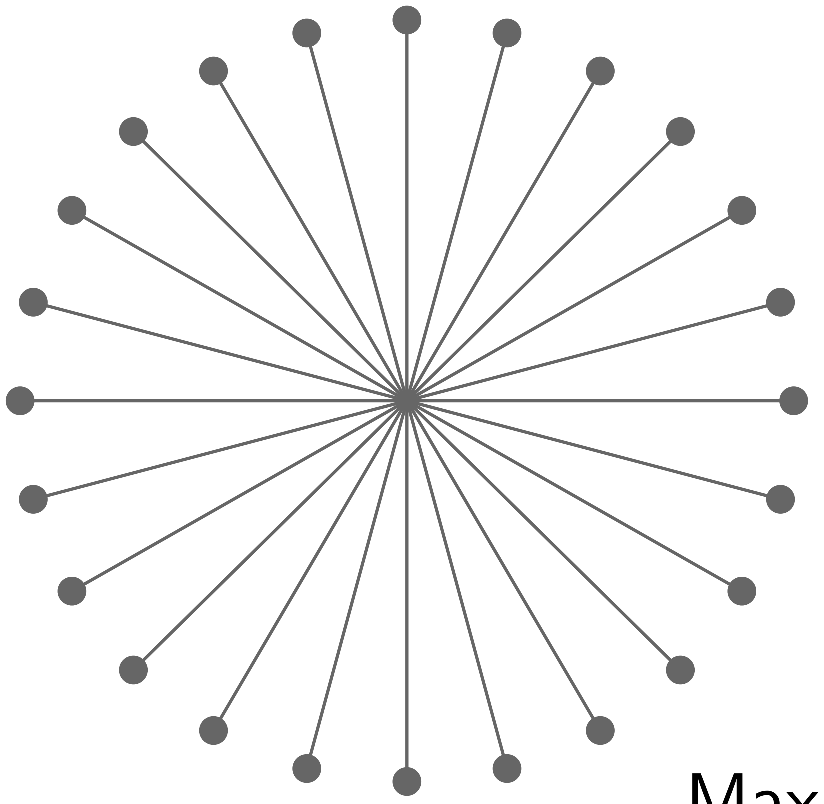
Straight-line ~~Bichromatic~~ Perfect Matchings with crossings

[Pach & Solymosi, 1999]

$$\mathrm{cr}(M_P^{\vee}) = \binom{n}{2}$$

$$= \frac{n^2 - n}{2}$$



$\longleftarrow$ P

Max-crossing matching, $M_P^{\vee}$

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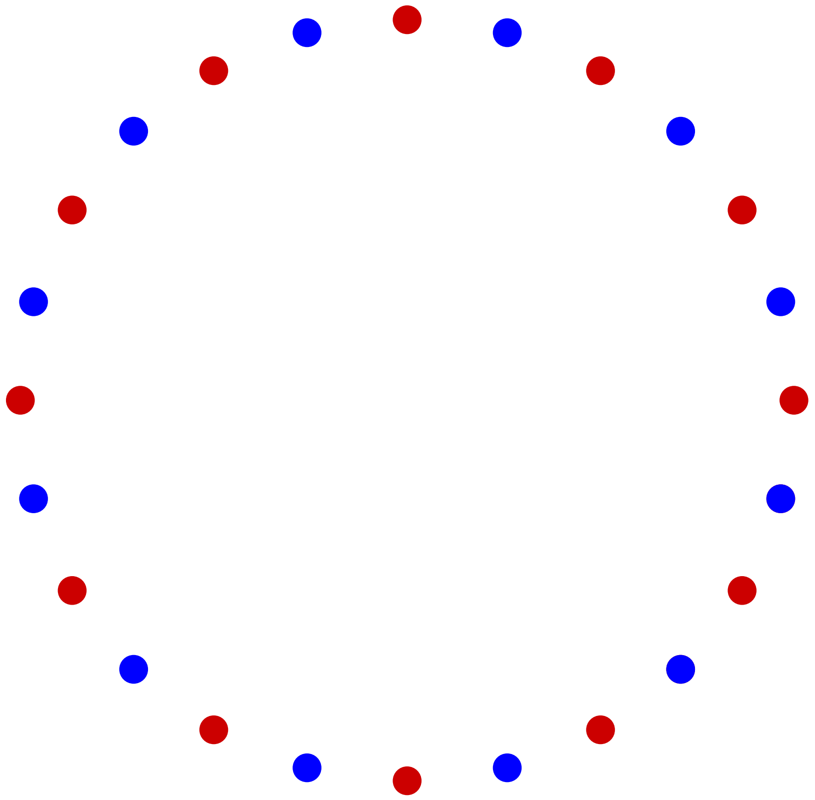

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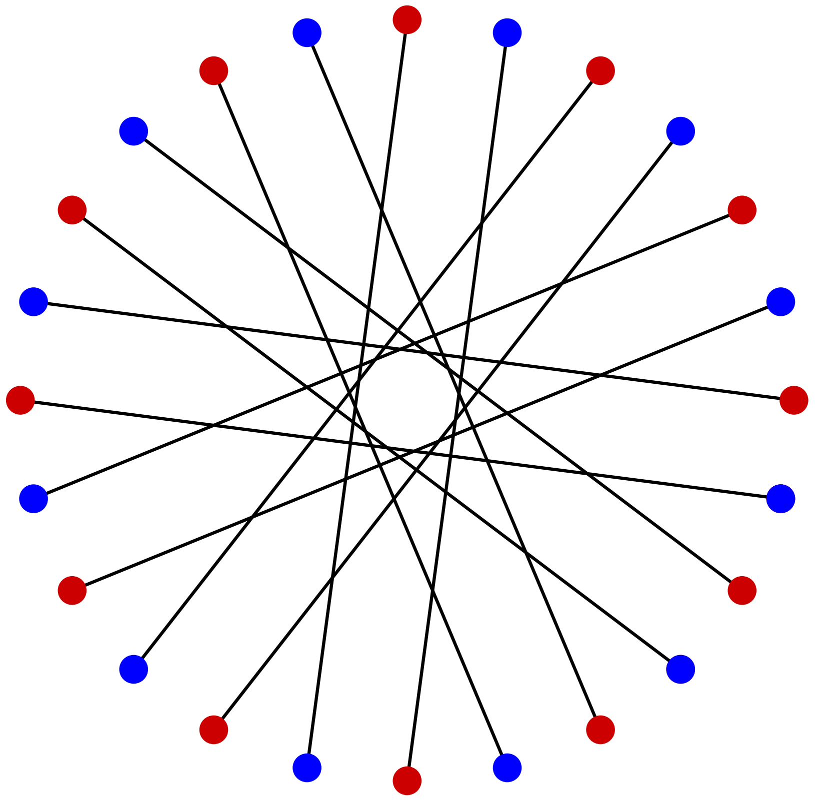

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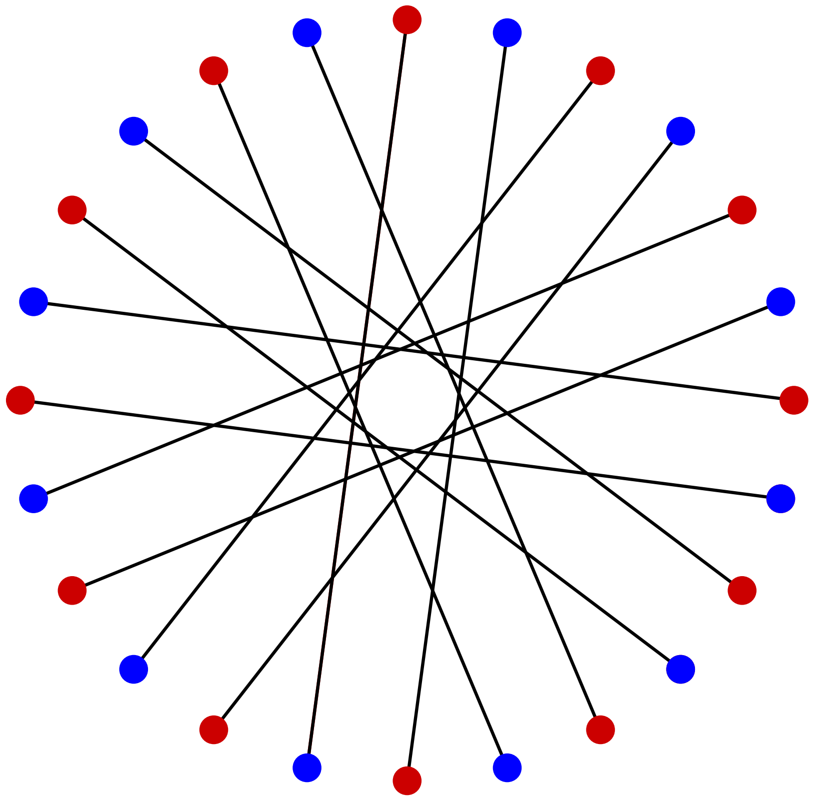

Total number of crossings $=$

$$\binom{n}{2} - \frac{n}{2}$$

Question

For which values of k does every bichromatic convex point set $P = R \cup B$, $|R| = |B| = n$, admit a max-crossing perfect matching ($M_P^{\vee}$) with at least k crossings?

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Step 1. Let Q be a convex point set with a special coloring. Then

$$\mathrm{cr}(M_Q^\vee) = \begin{cases} \frac{3n^2}{8} - \frac{n}{2} & \text{if } n \equiv 0 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 1 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} - \frac{1}{2} & \text{if } n \equiv 2 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

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$$\frac{3n^2}{8} - \frac{n}{2} + c$$

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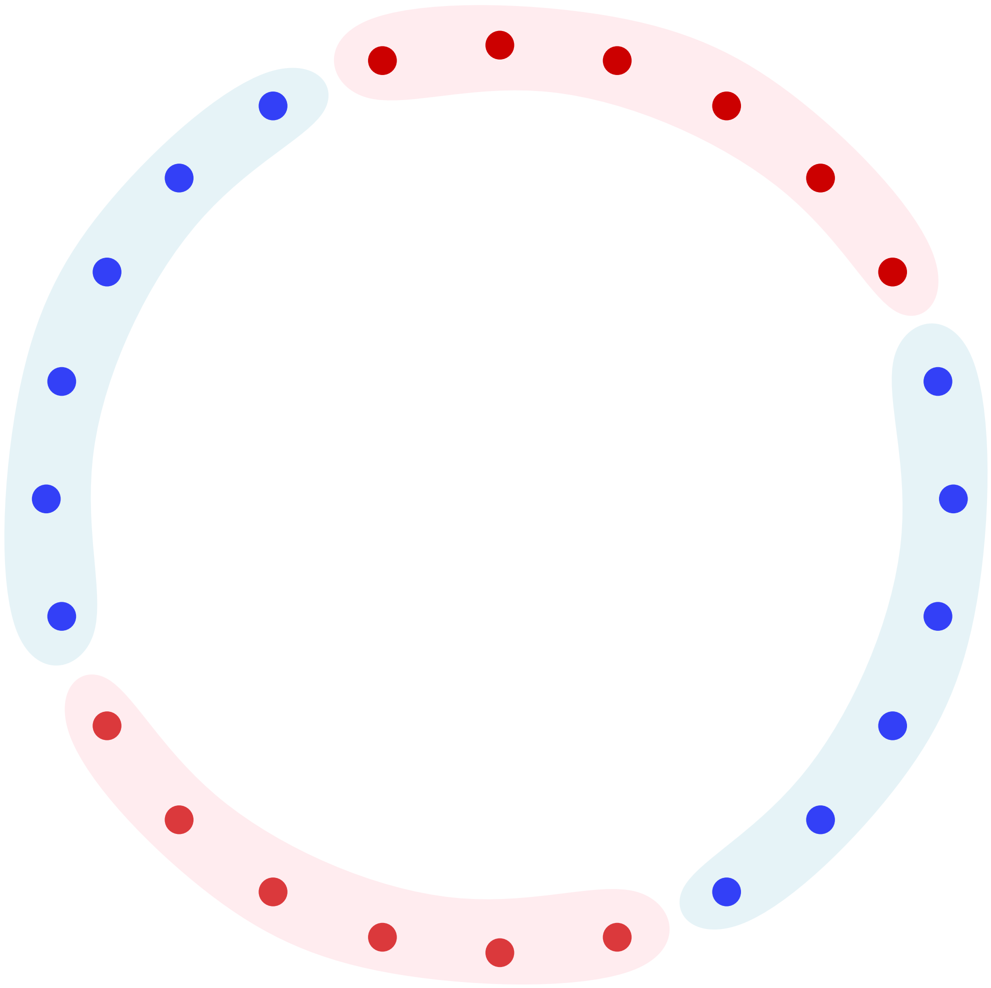

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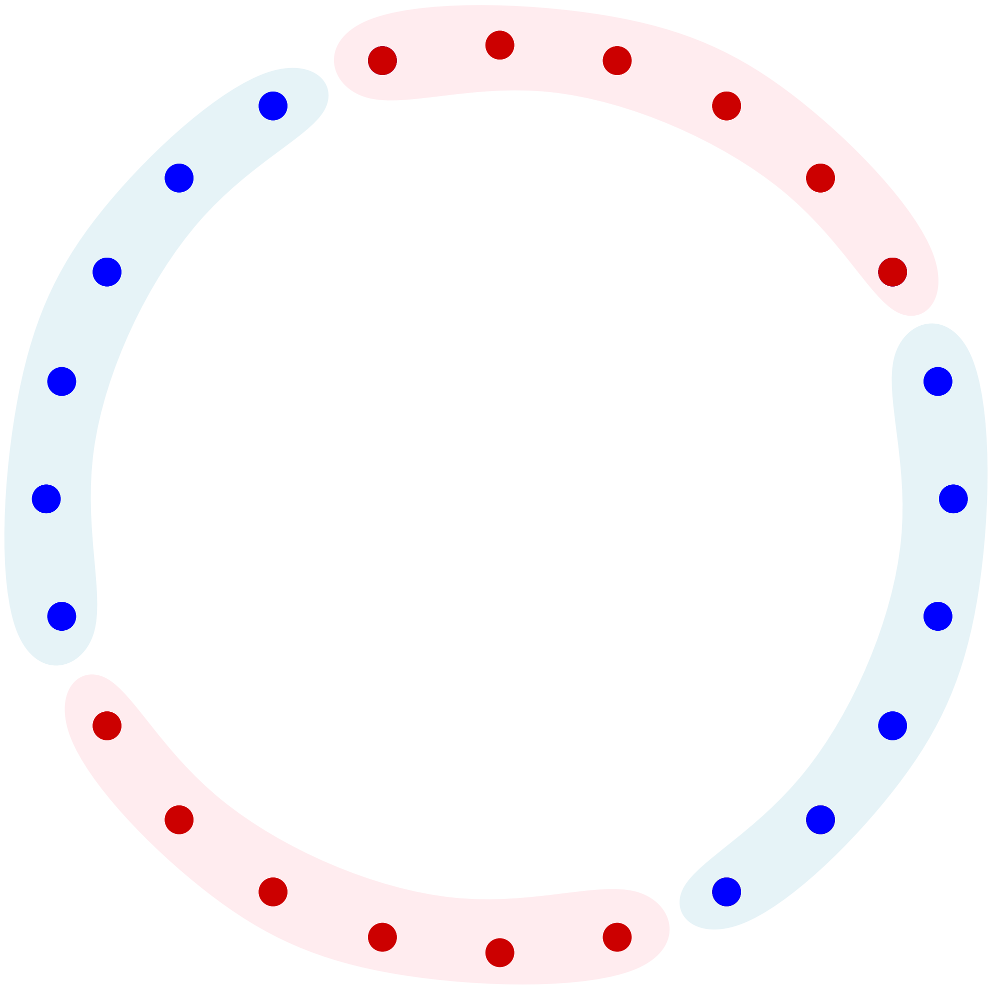

Balanced 4 - block coloring Q

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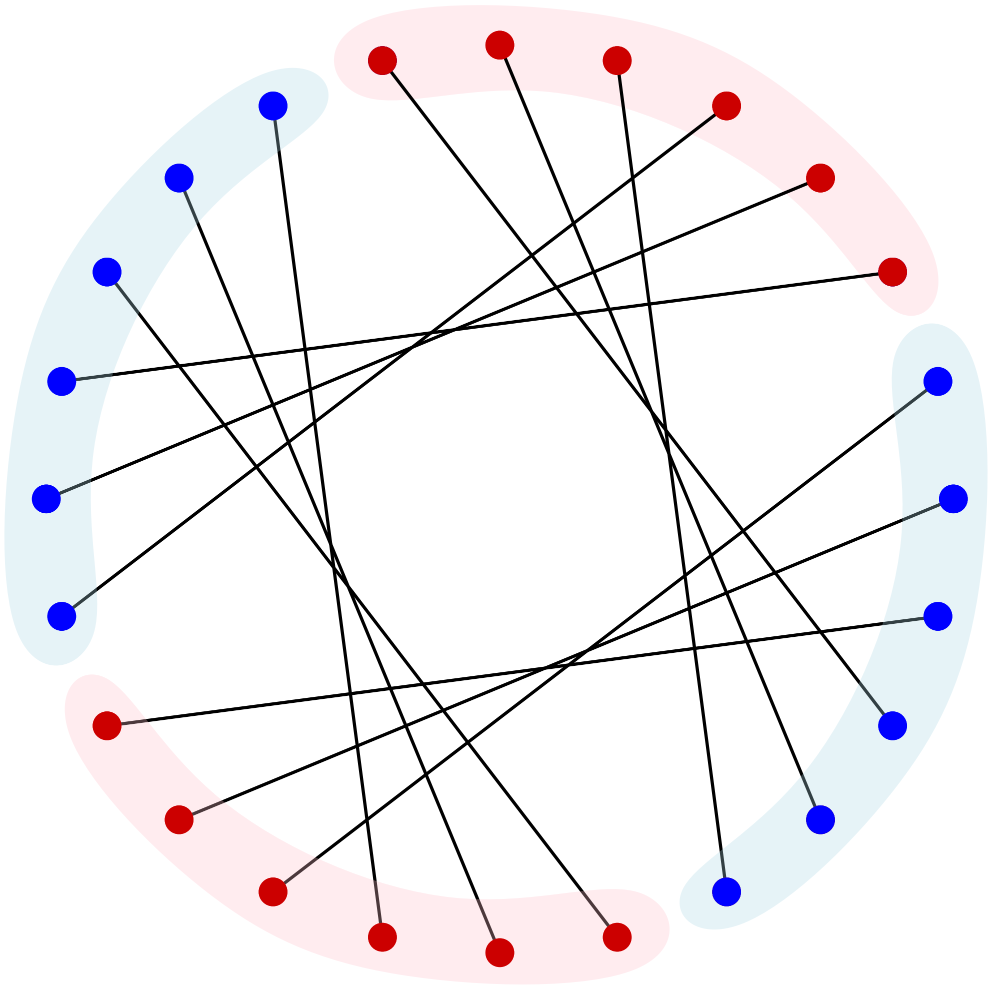

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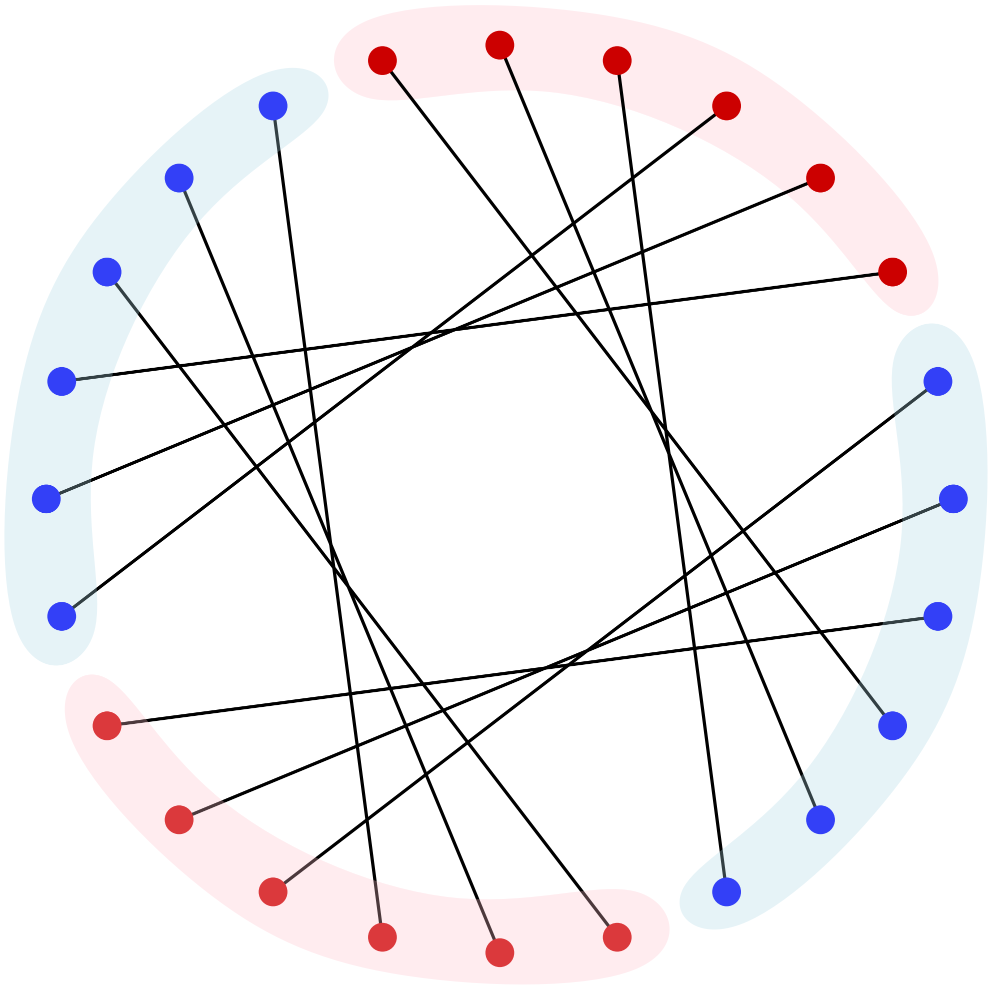

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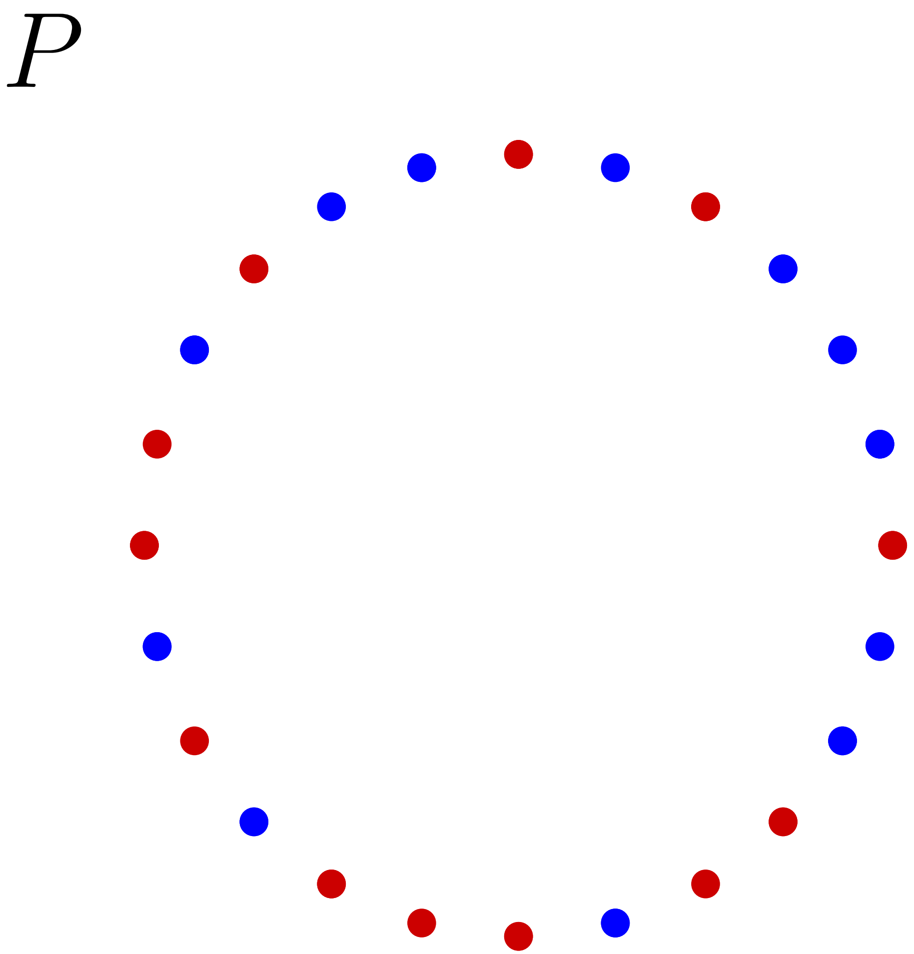

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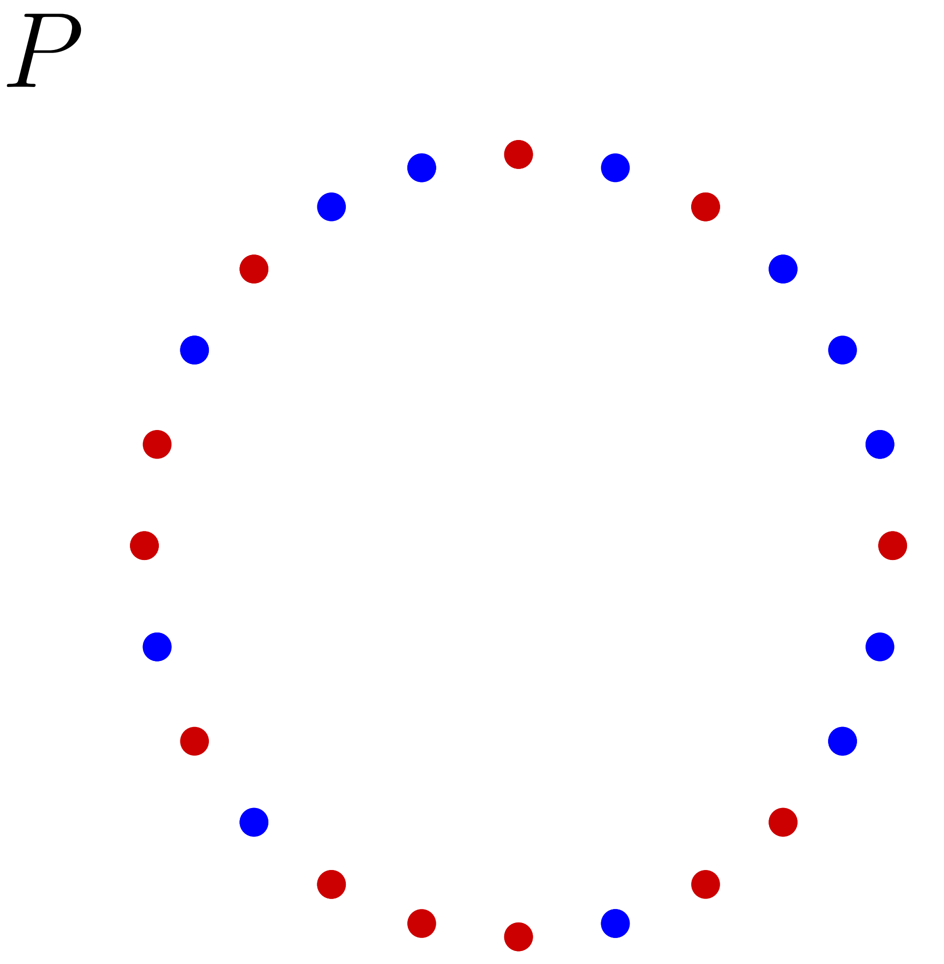

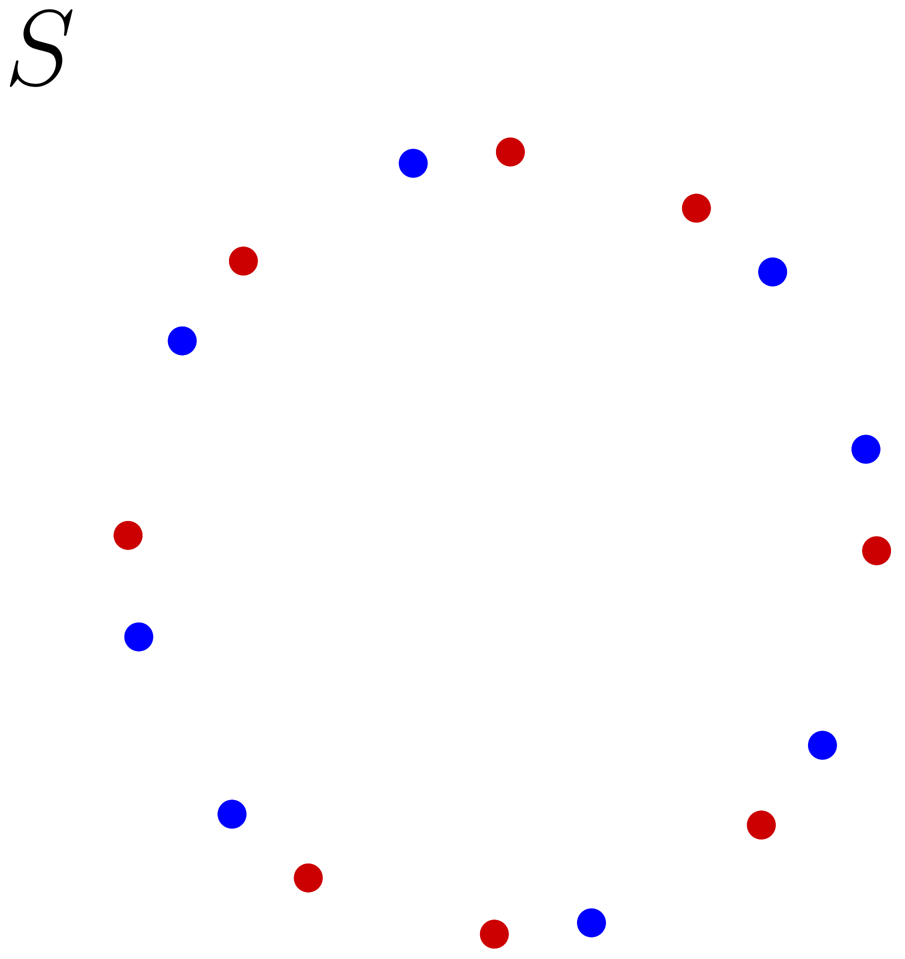

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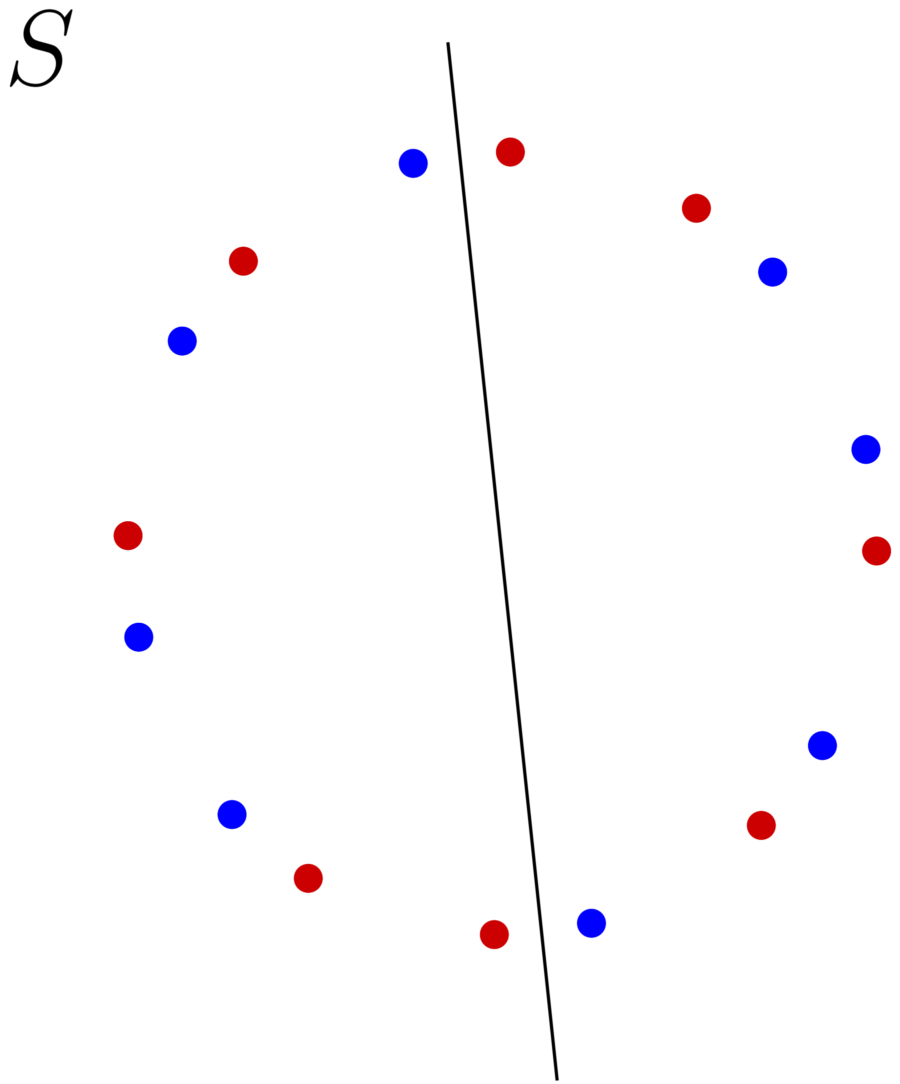

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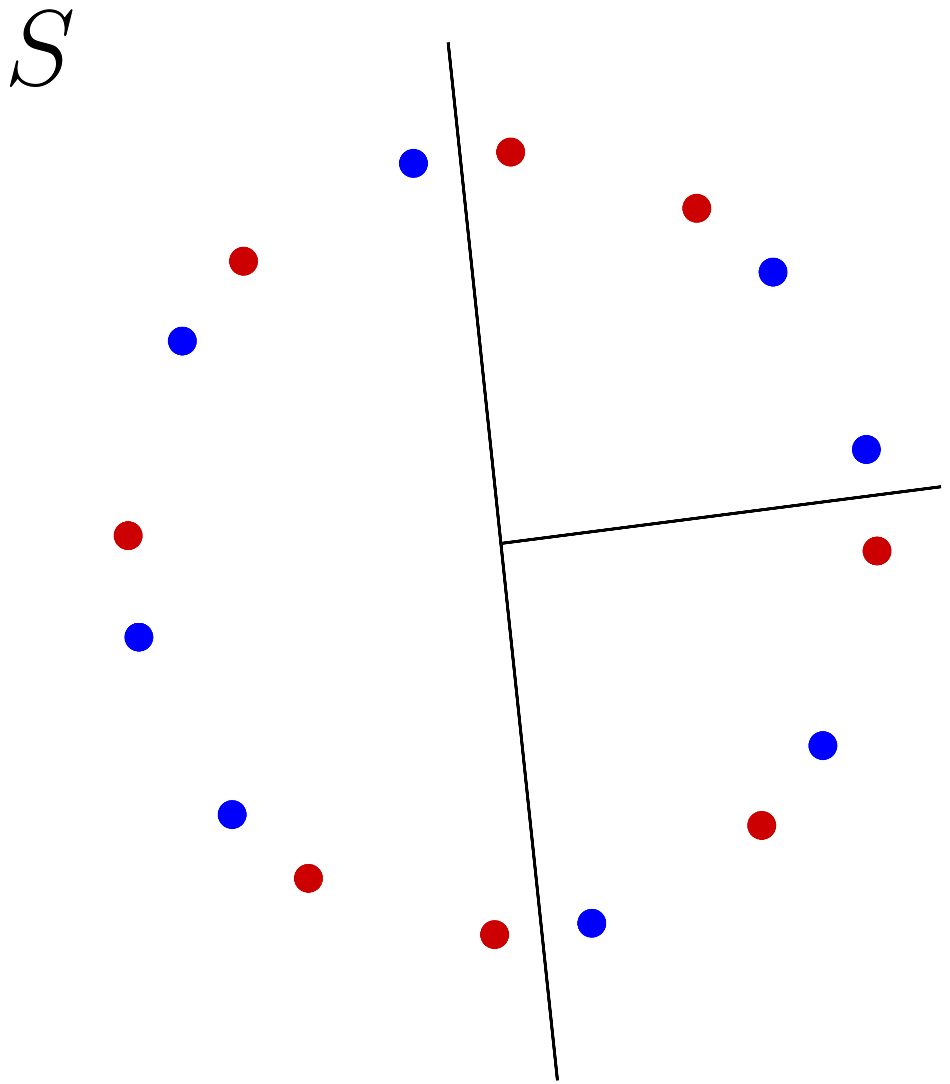

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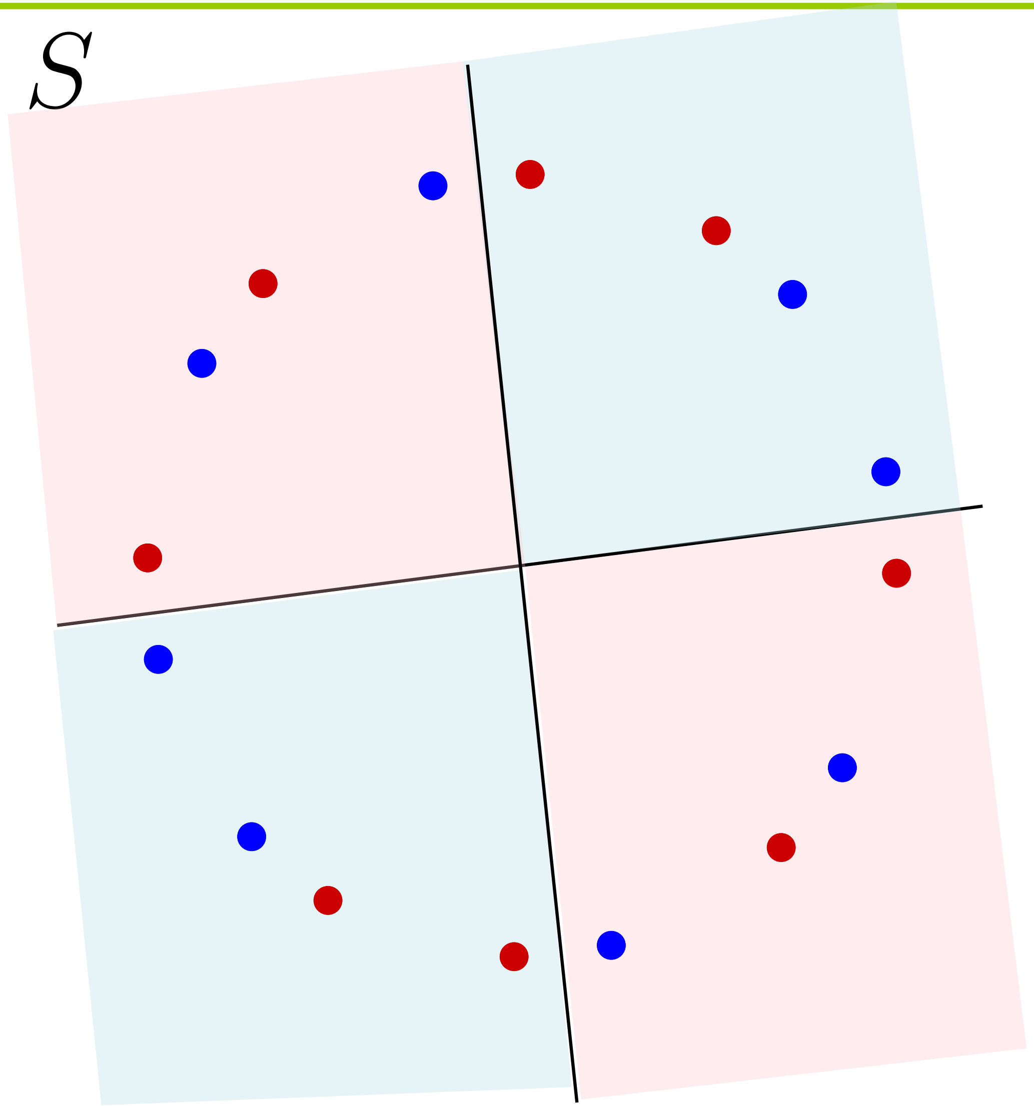

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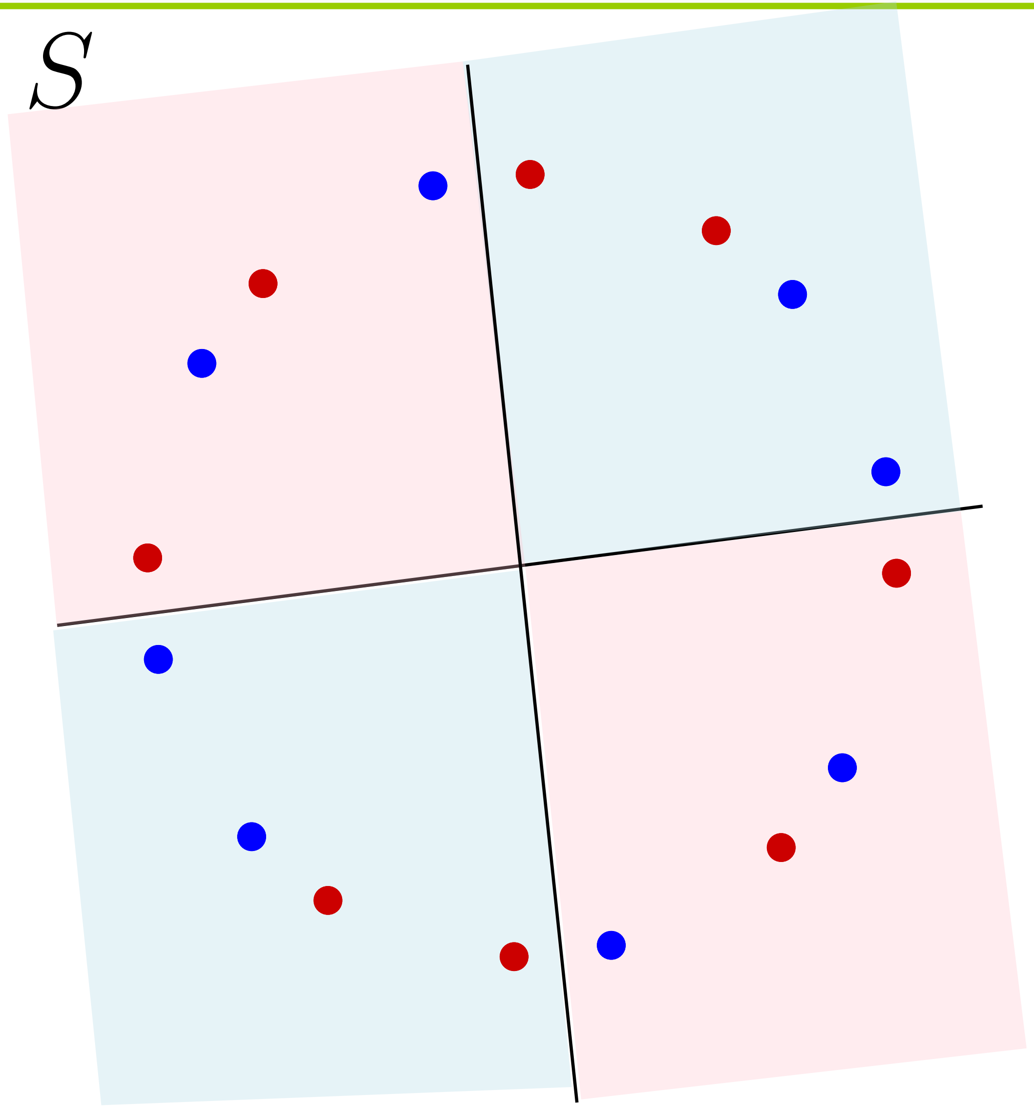

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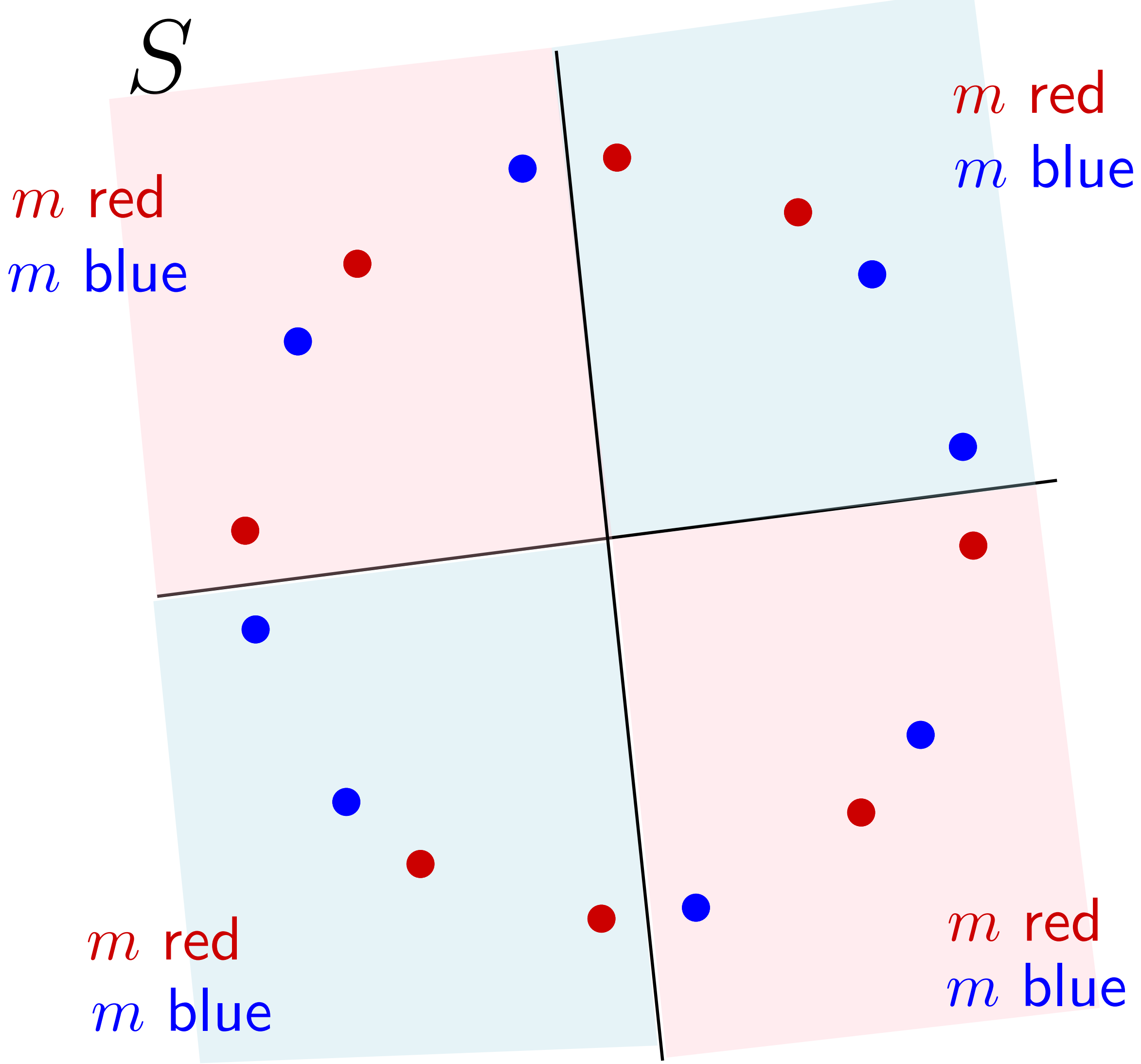

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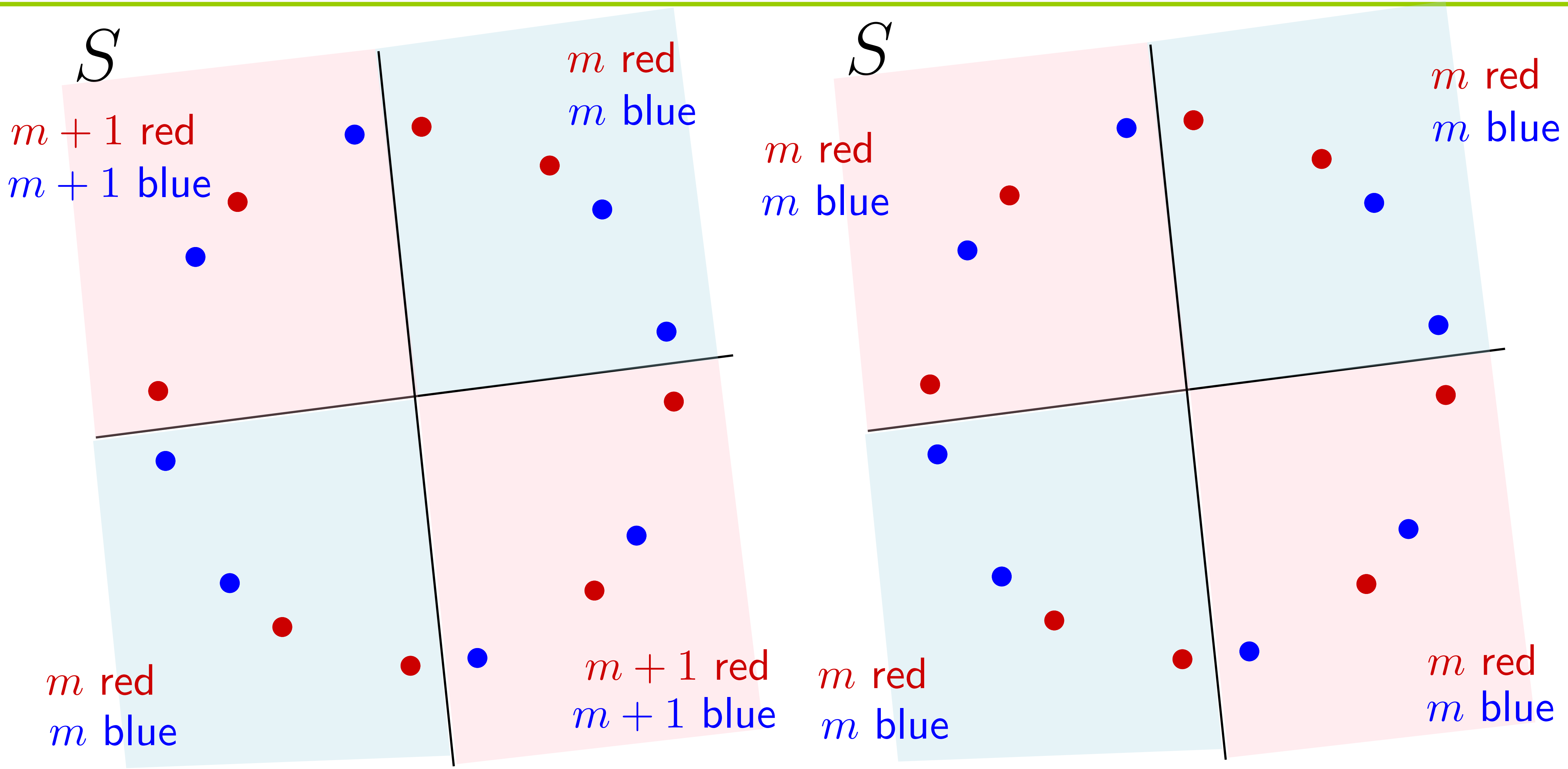

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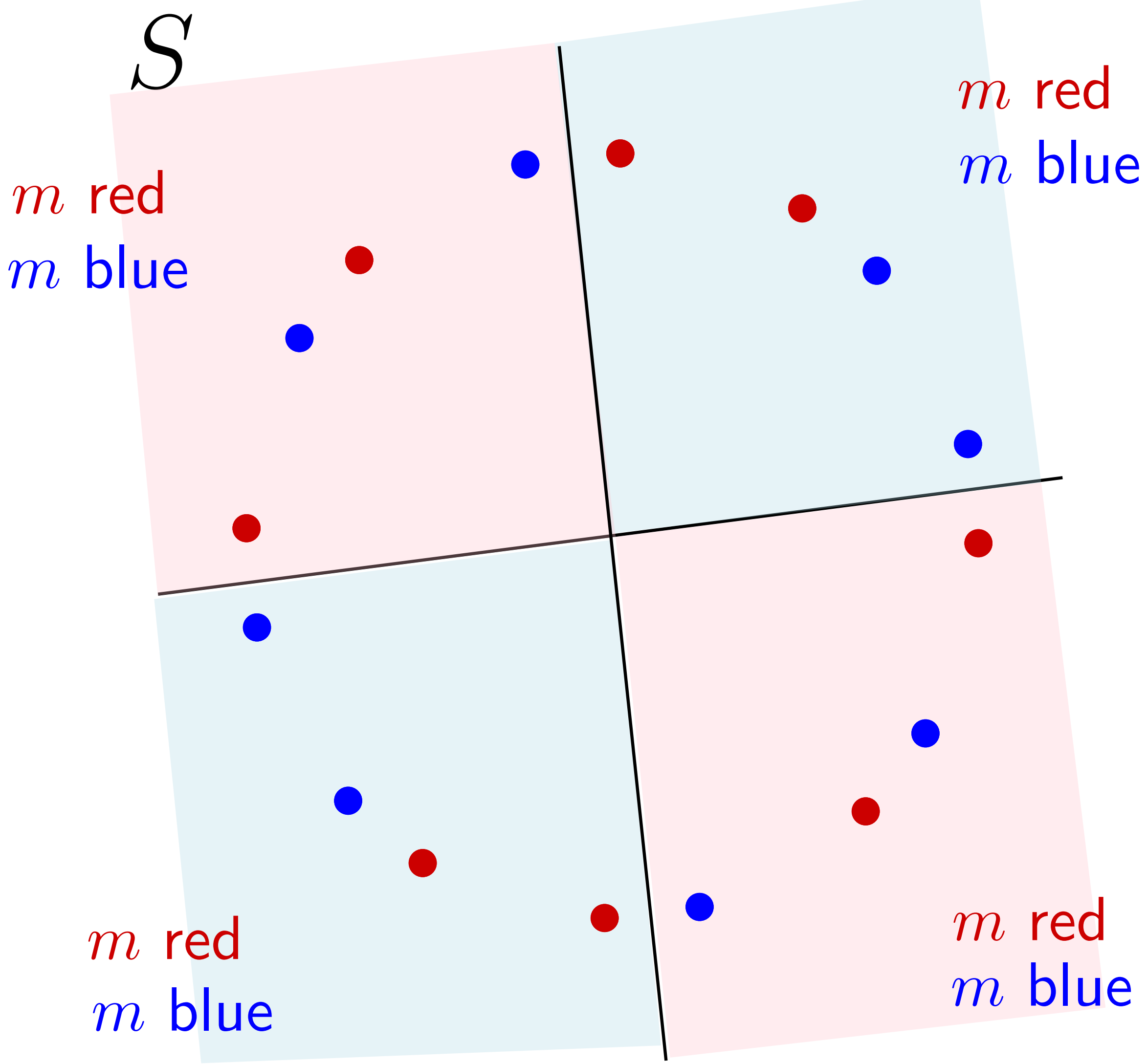

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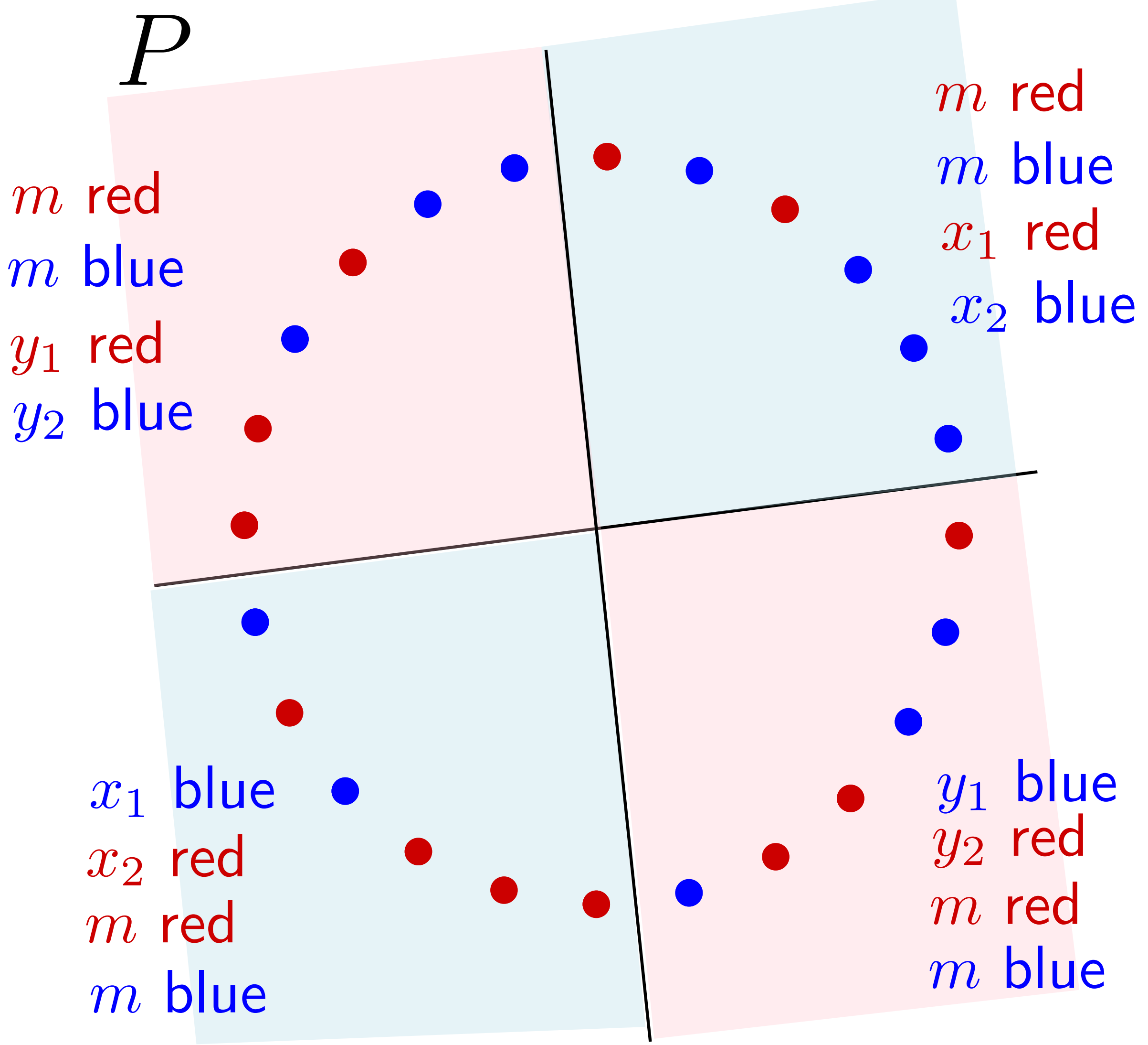

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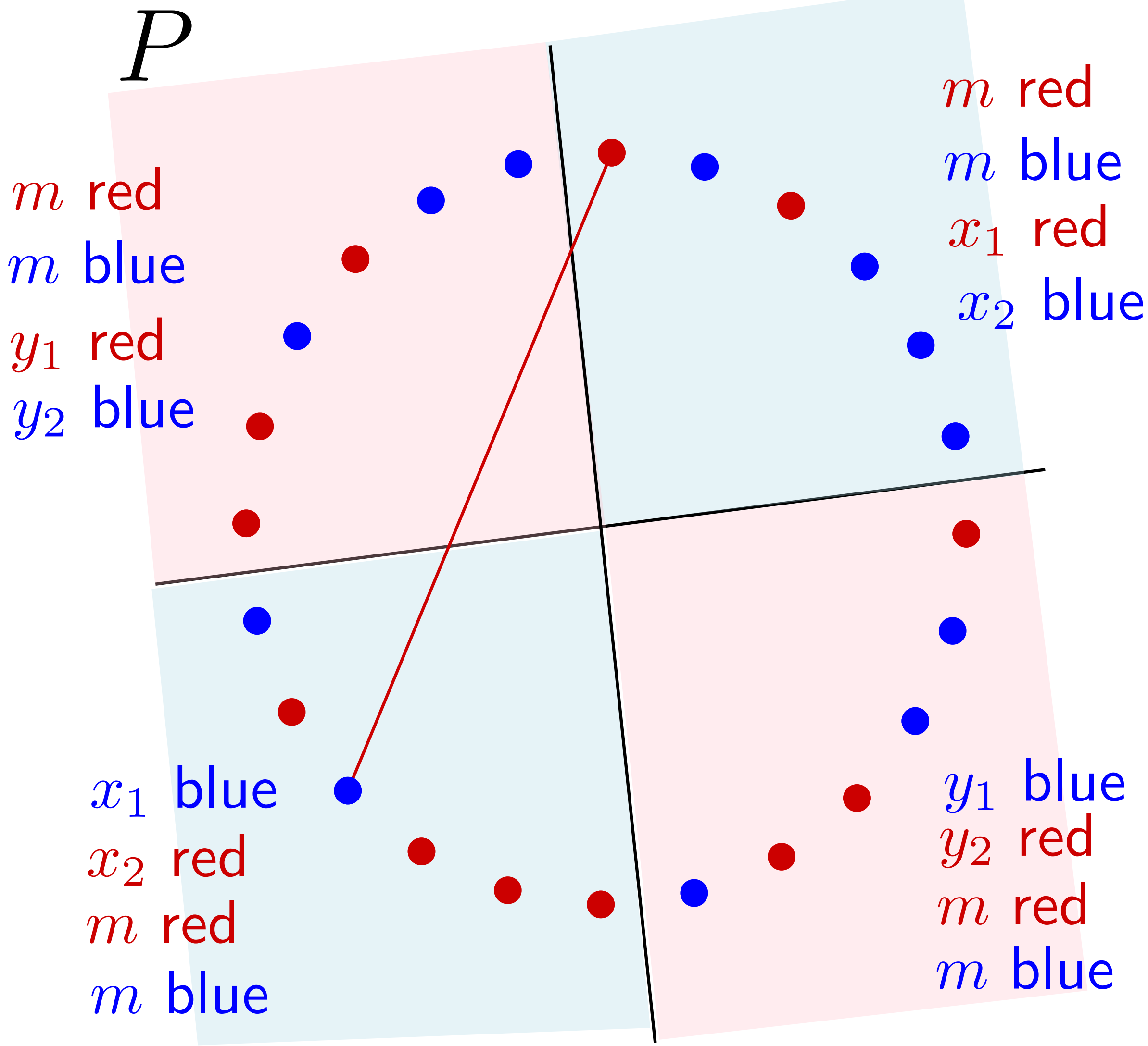

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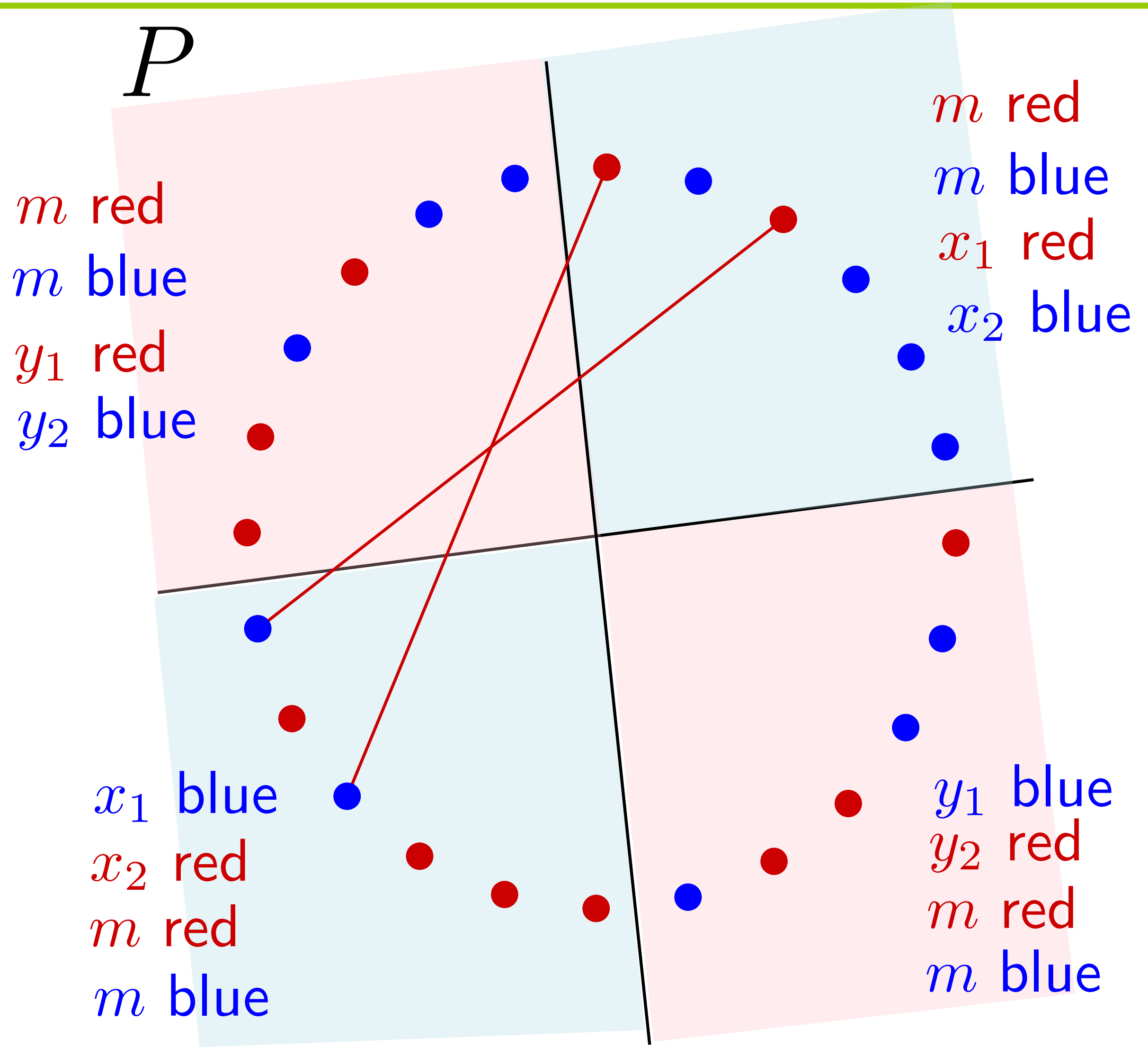

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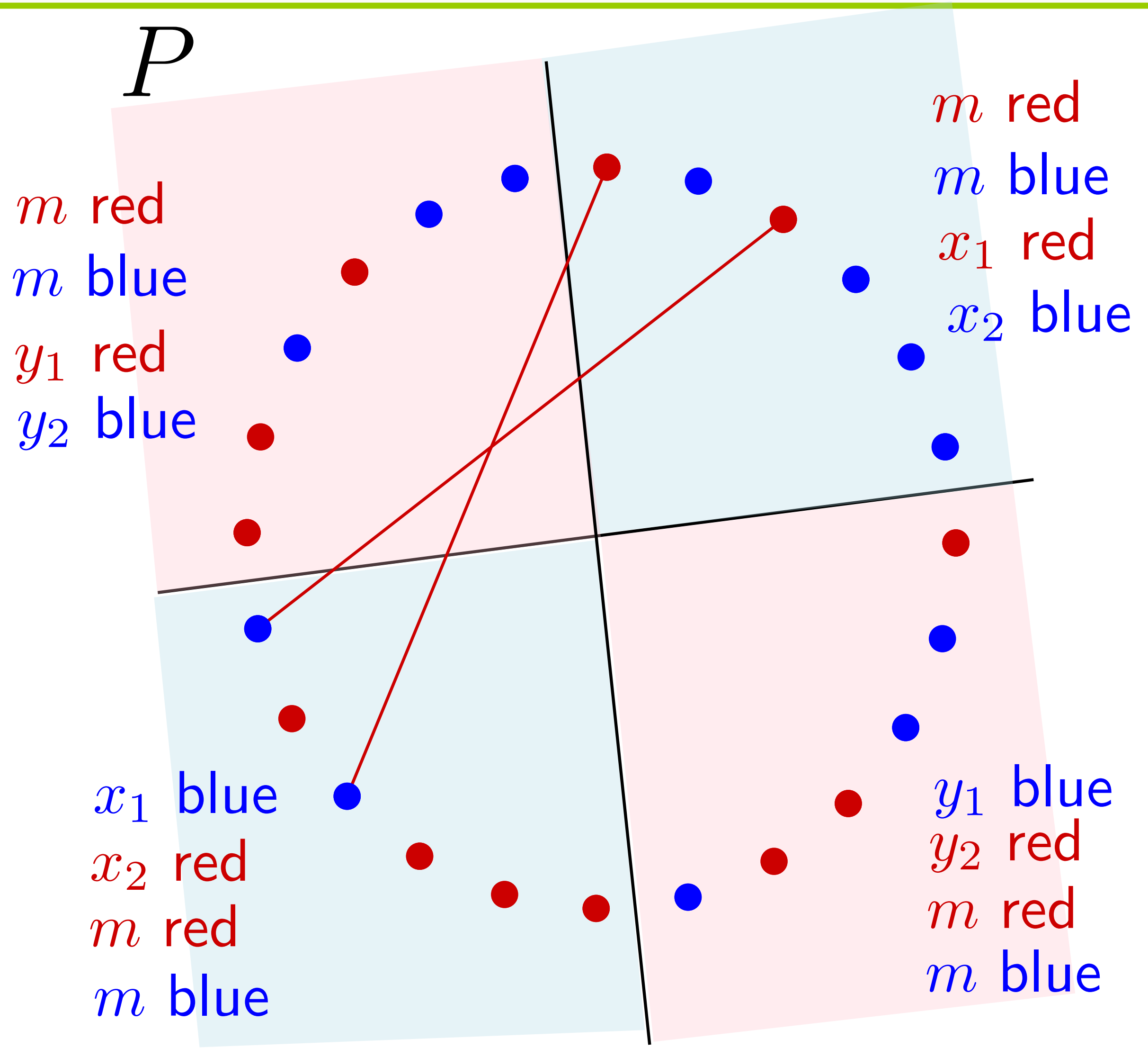
$\binom{m+x_1}{2}$



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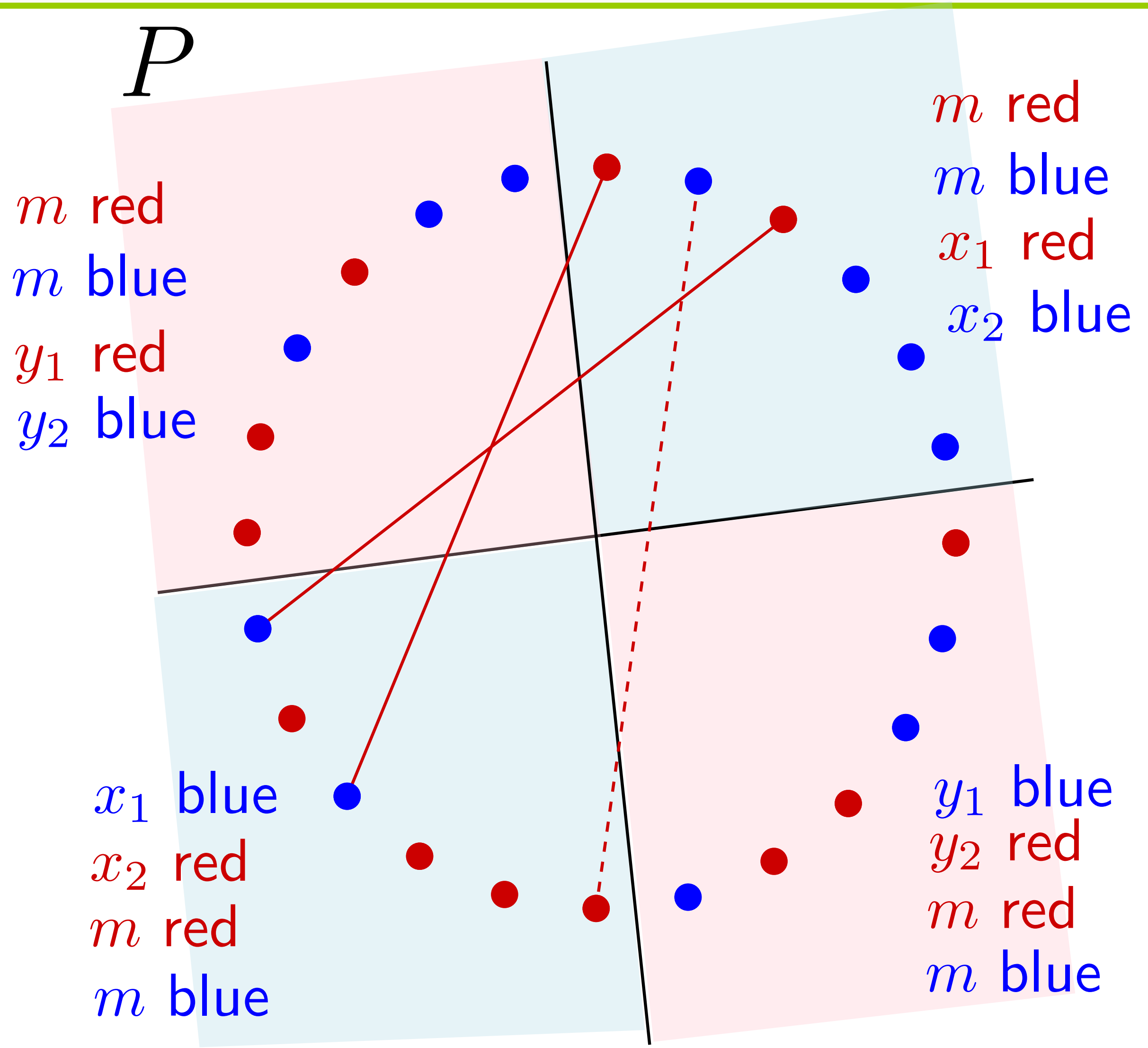

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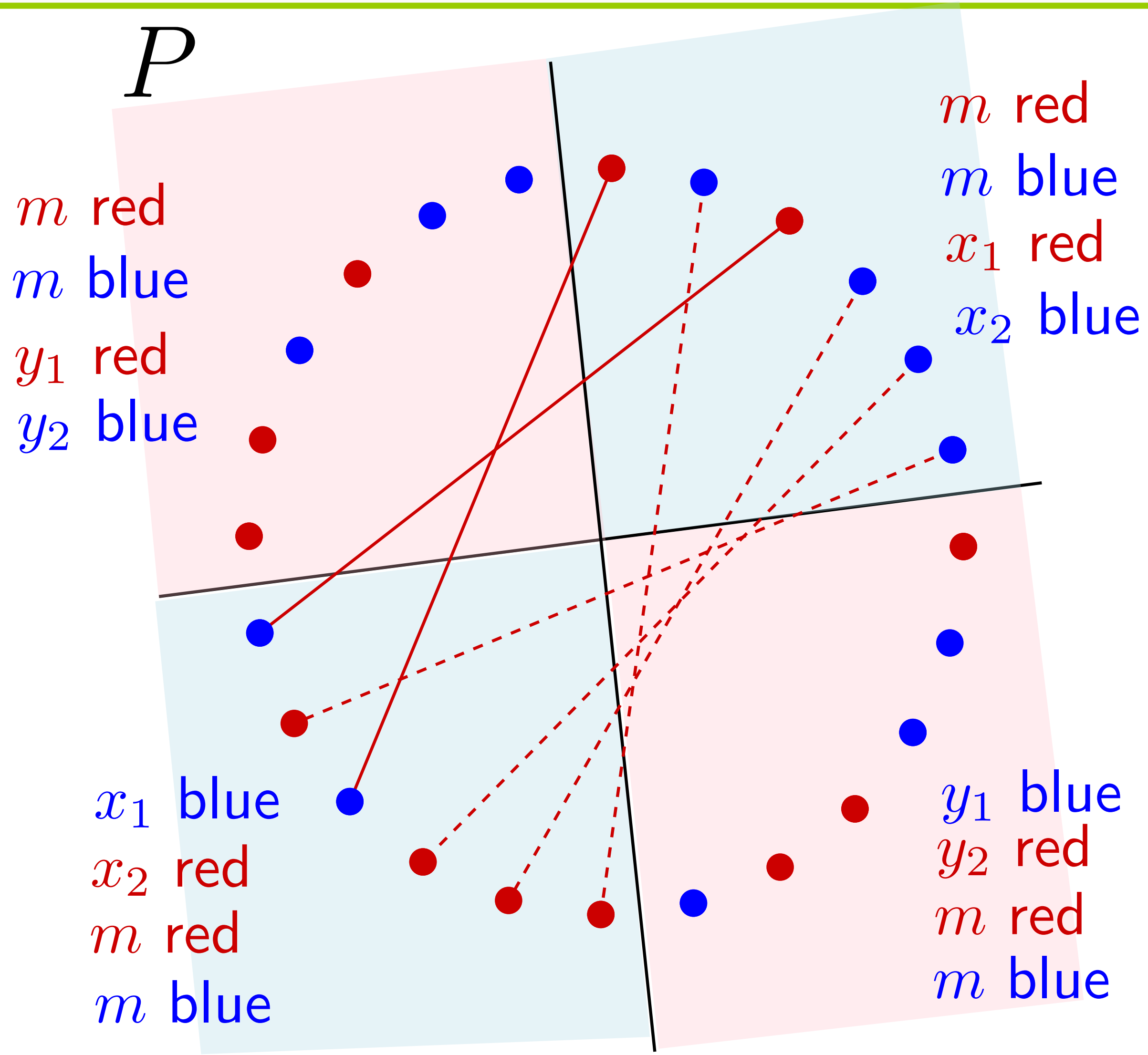

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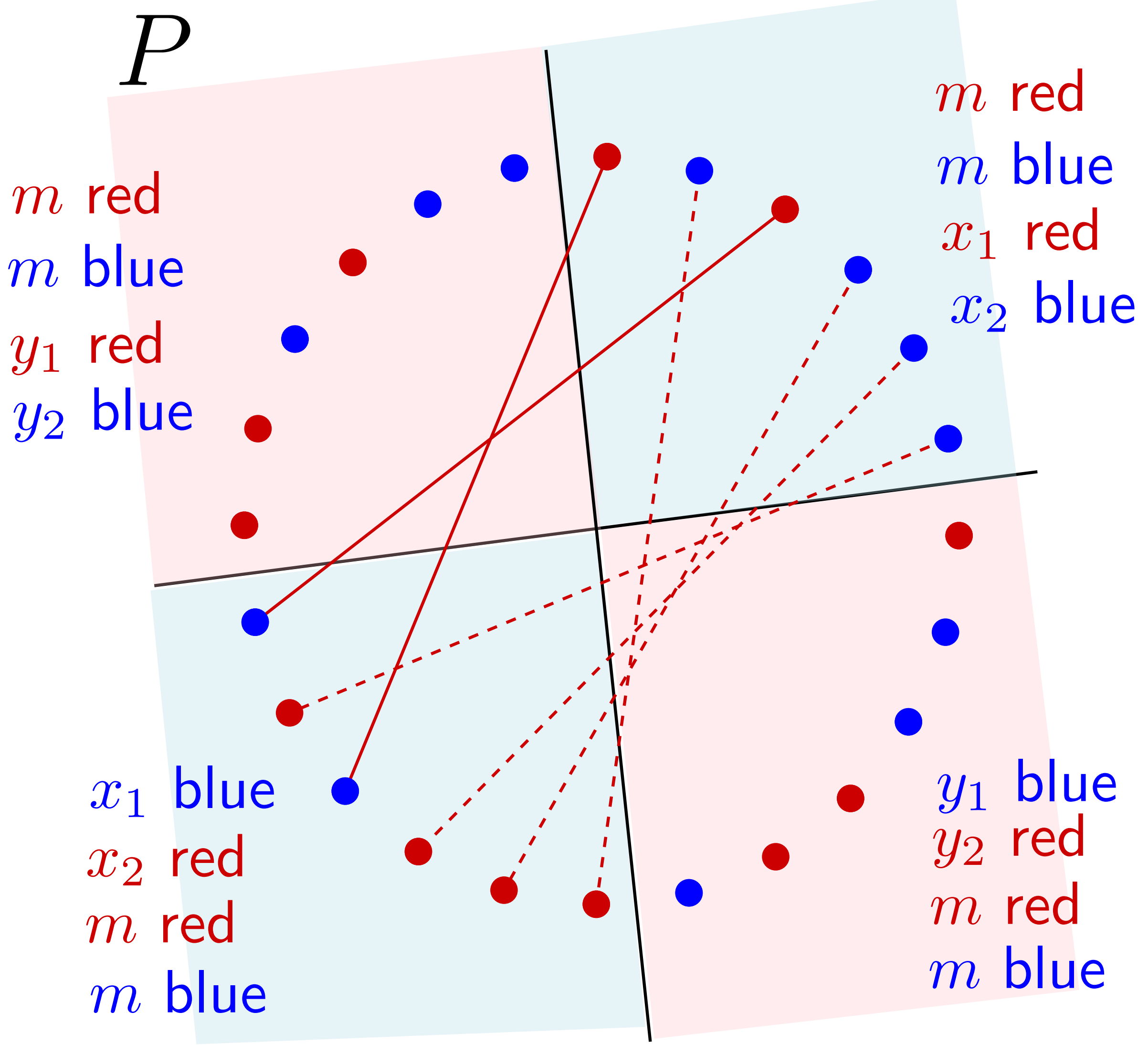
$$\binom{m+x_1}{2} + \binom{m+x_2}{2}$$



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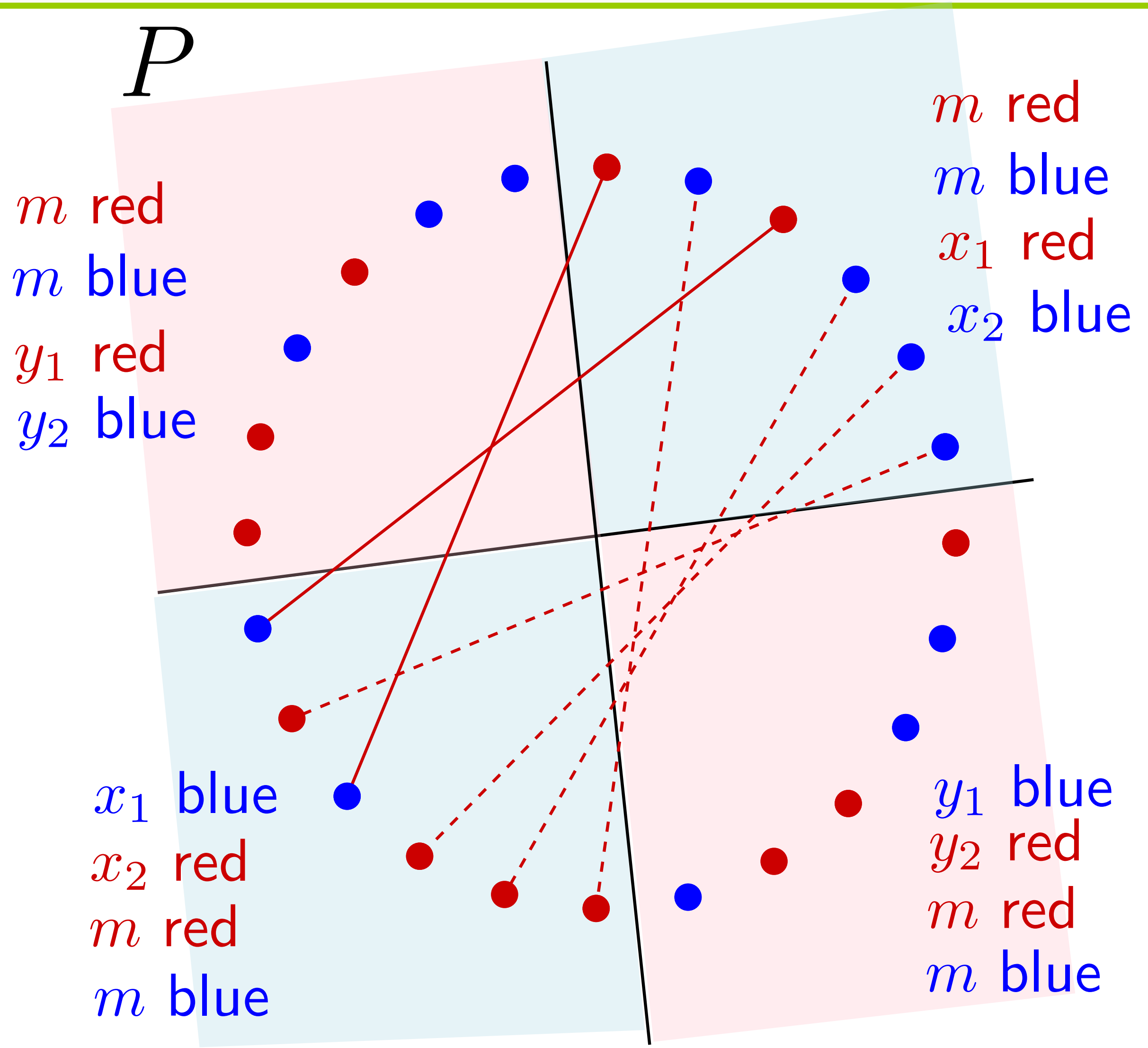
$$\geq \binom{m+x_1}{2} + \binom{m+x_2}{2} + x_1 x_2$$



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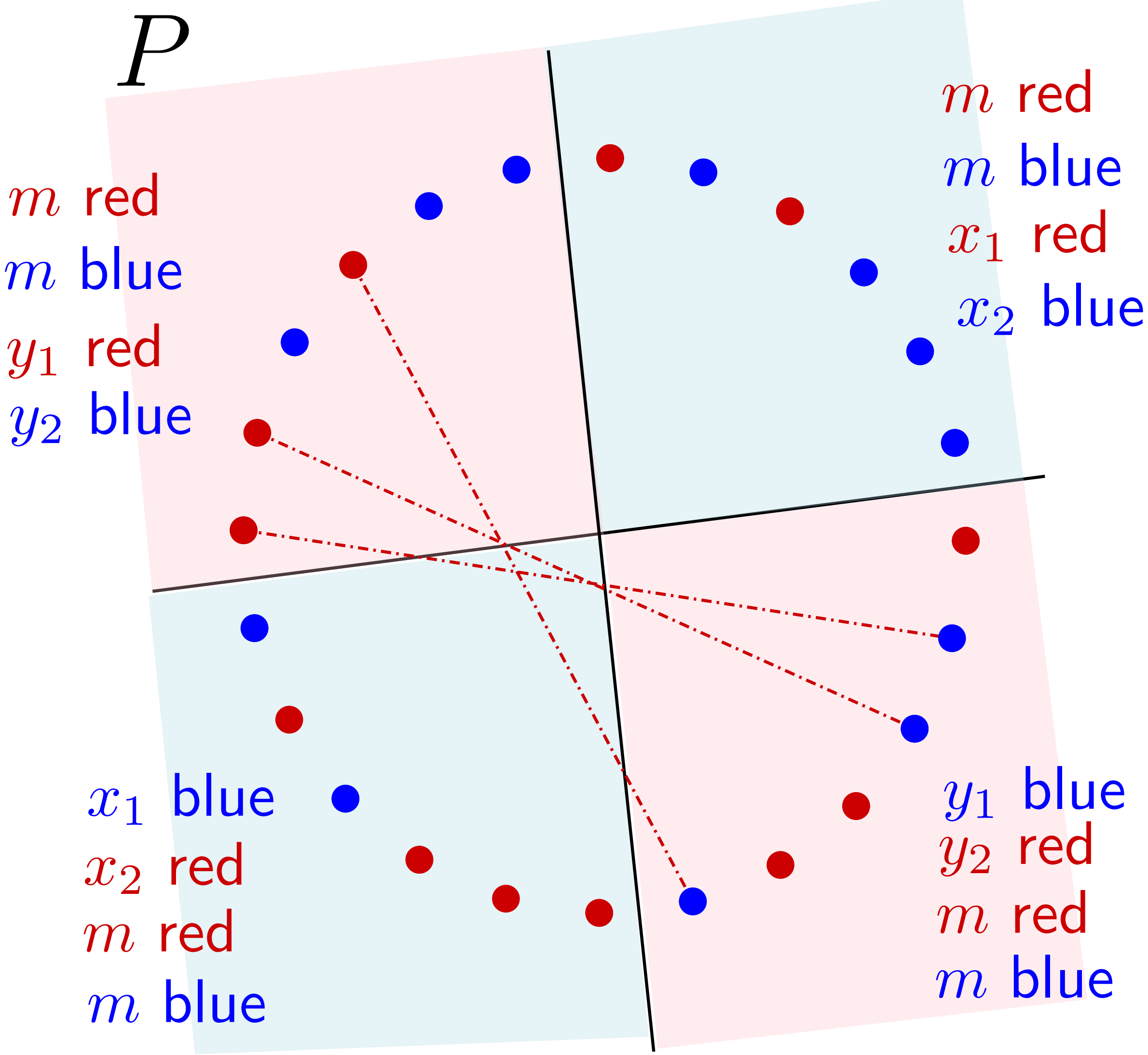

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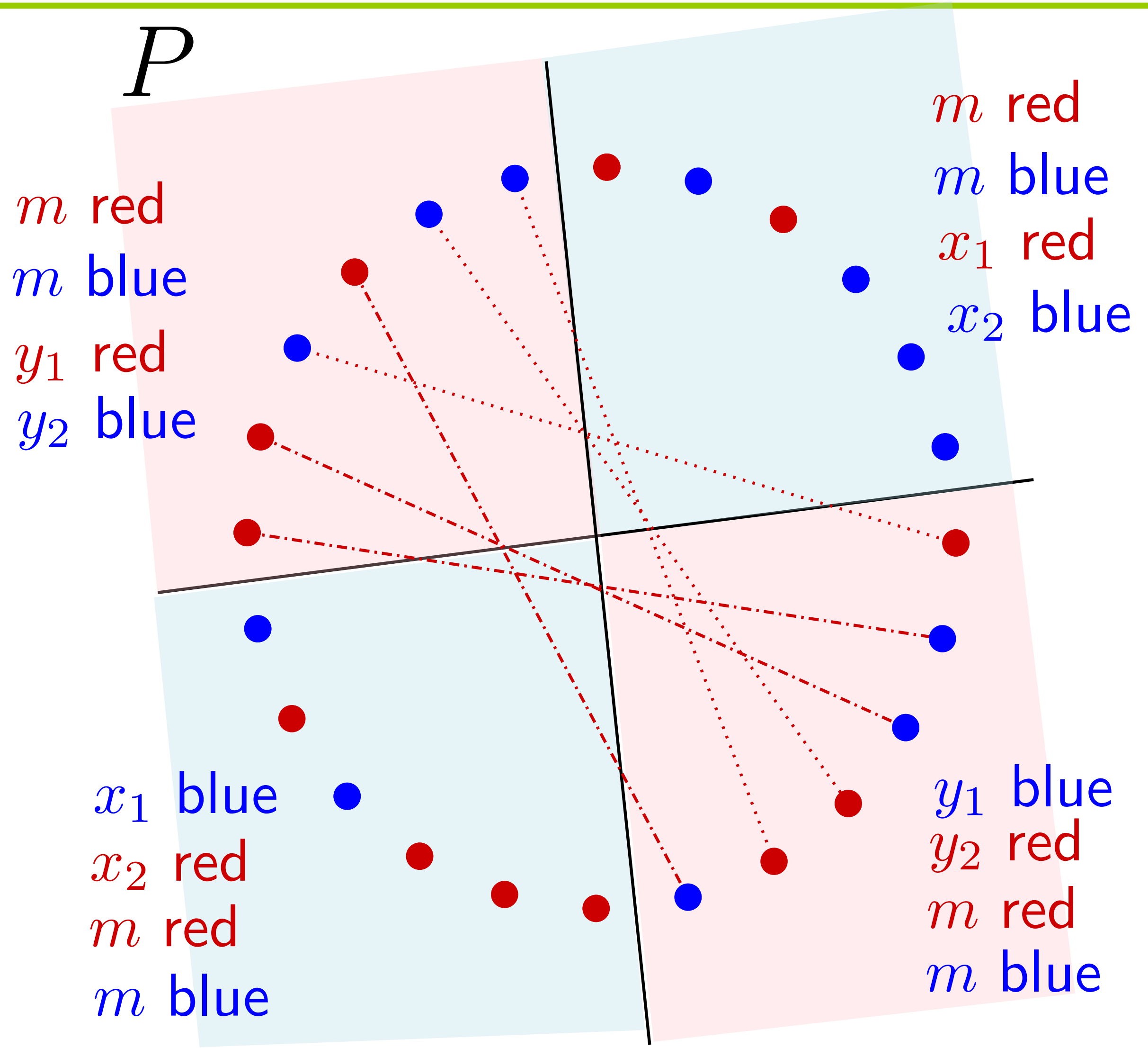

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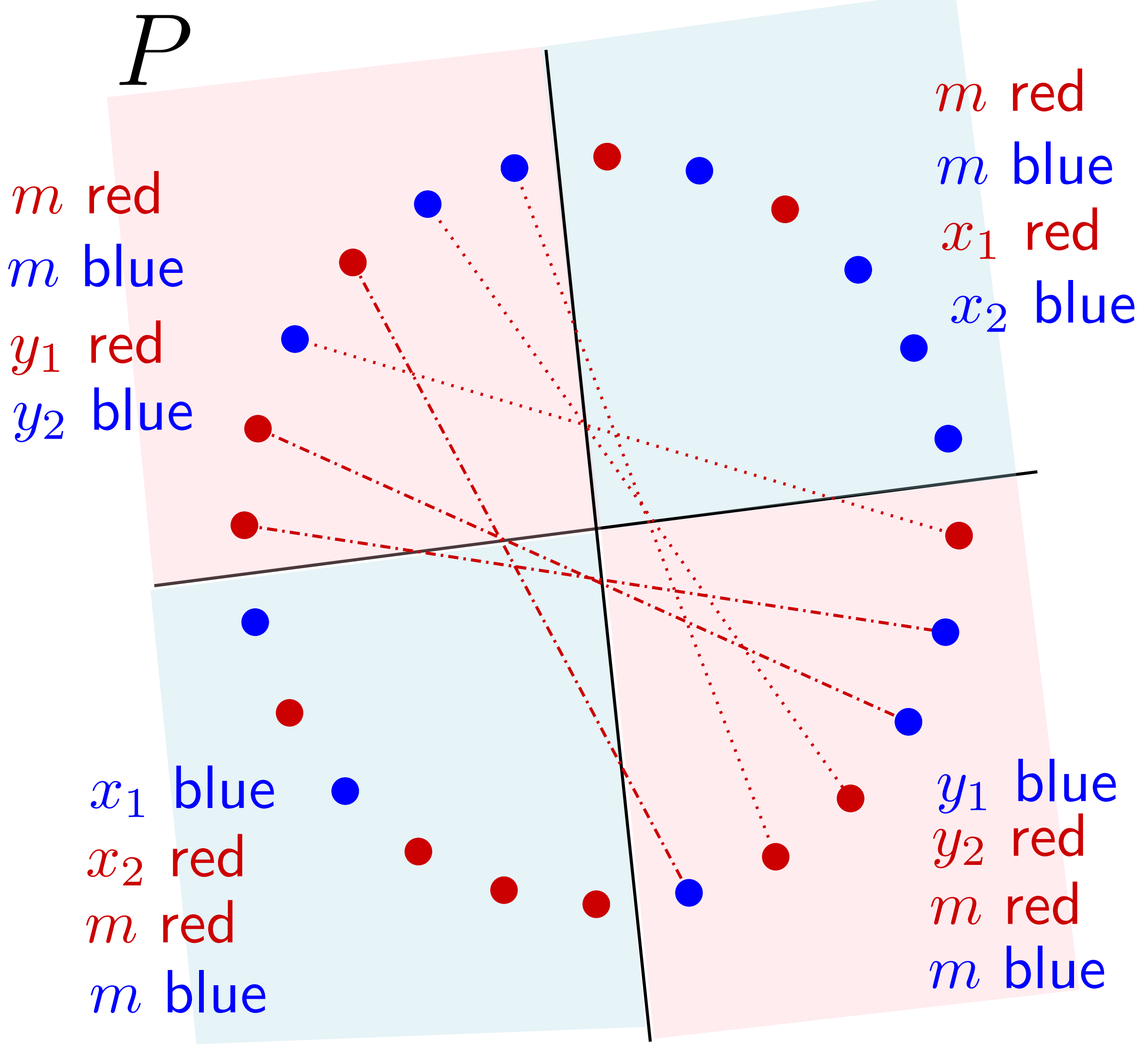

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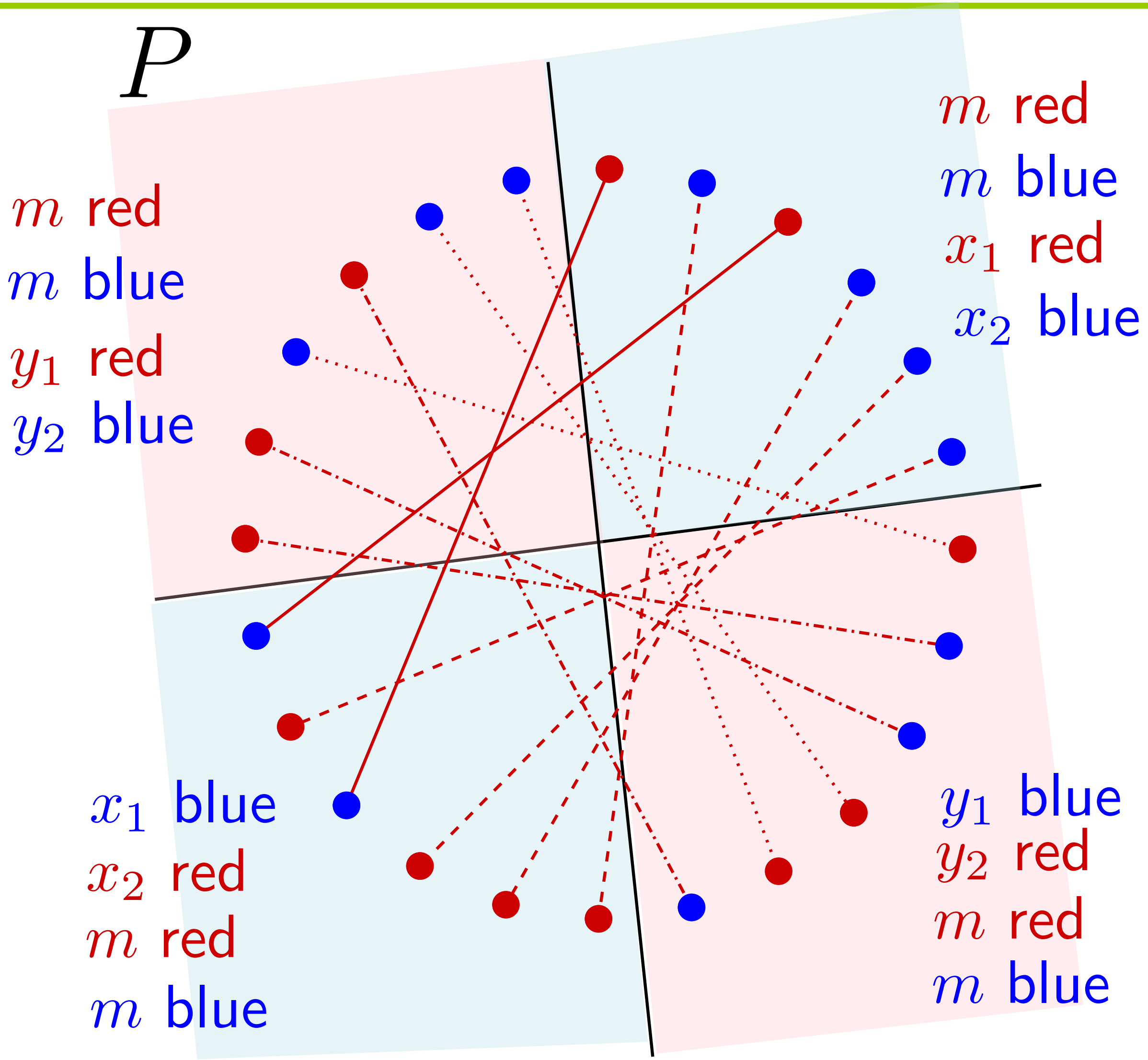

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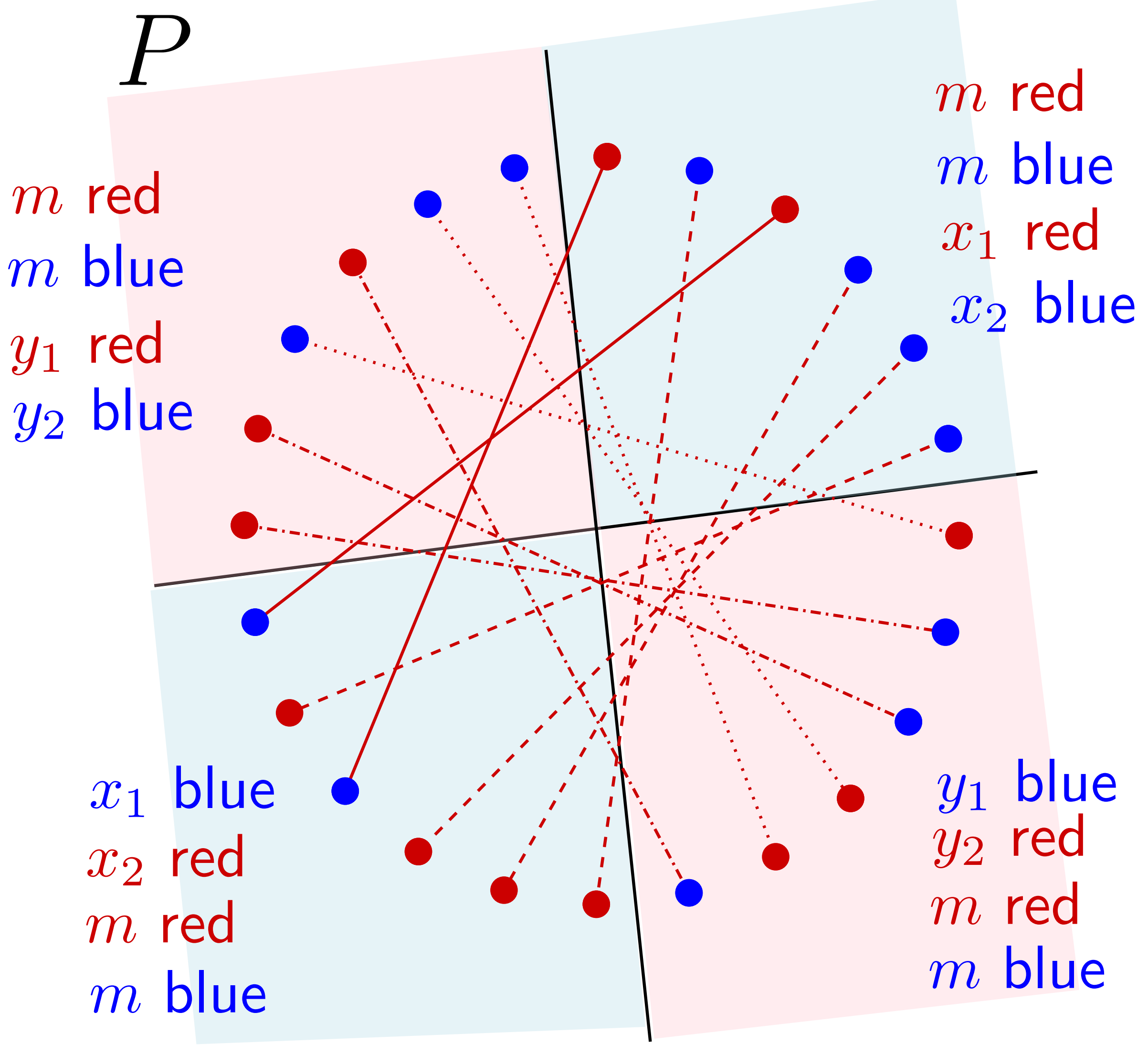
$$\mathrm{cr}(M_P) \geq \binom{m+x_1}{2} + \binom{m+x_2}{2} + x_1 x_2 + \binom{m+y_1}{2} + \binom{m+y_2}{2} + y_1 y_2 + (2m + x_1 + x_2)(2m + y_1 + y_2)$$



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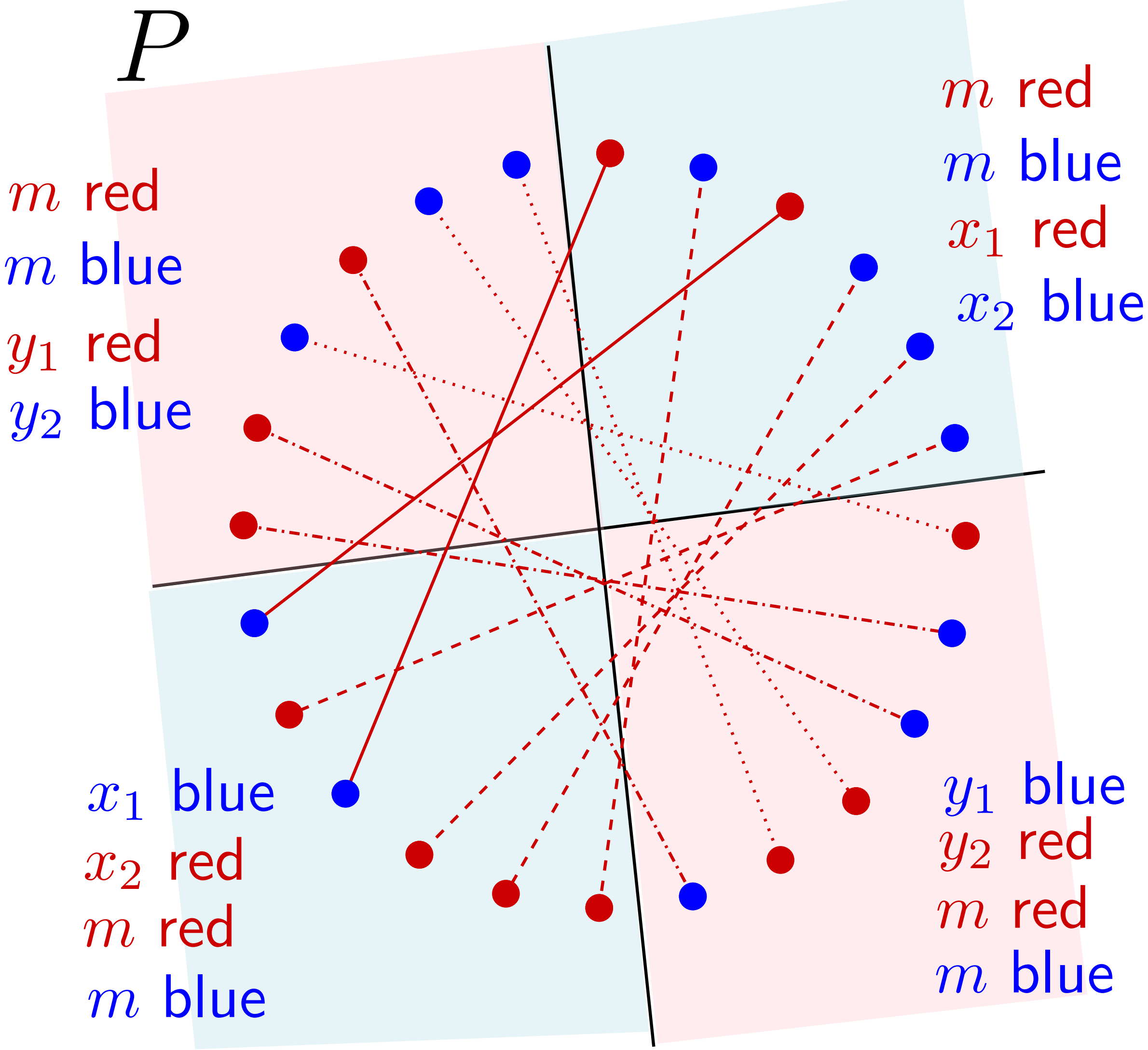
$$\geq 6m^2 + 3m(x_1+x_2+y_1+y_2) - 2m + x_1y_1 + x_1y_2 + x_2y_1 + x_1x_2 + y_1y_2 + x_2y_2 + \frac{x_2^2+x_1^2+y_1^2+y_2^2-x_1-x_2-y_1-y_2}{2}$$



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$$= 6m^2 + 3m(x_1 + x_2 + y_1 + y_2) + \frac{3(x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2)}{4}$$

$$+ \frac{3(x_1x_2 + y_1y_2}{4} + \frac{3(x_1^2 + x_2^2)}{8} + \frac{3(y_1^2 + y_2^2)}{8} - 2m - \frac{x_1}{2} - \frac{x_2}{2} - \frac{y_1}{2} - \frac{y_2}{2} + c$$

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$$c \leq \frac{1}{8}$$

Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.

$$\mathrm{cr}(M_P) - \mathrm{cr}(M_Q^\vee) \geq \frac{(x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2)}{4} + \frac{(x_1^2 + x_2^2)}{8} + \frac{(y_1^2 + y_2^2)}{8} + \frac{(x_1 x_2 + y_1 y_2)}{4} - c.$$

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As $S \subset P$, at least one among x_1, x_2, y_1, y_2 is at least 1.

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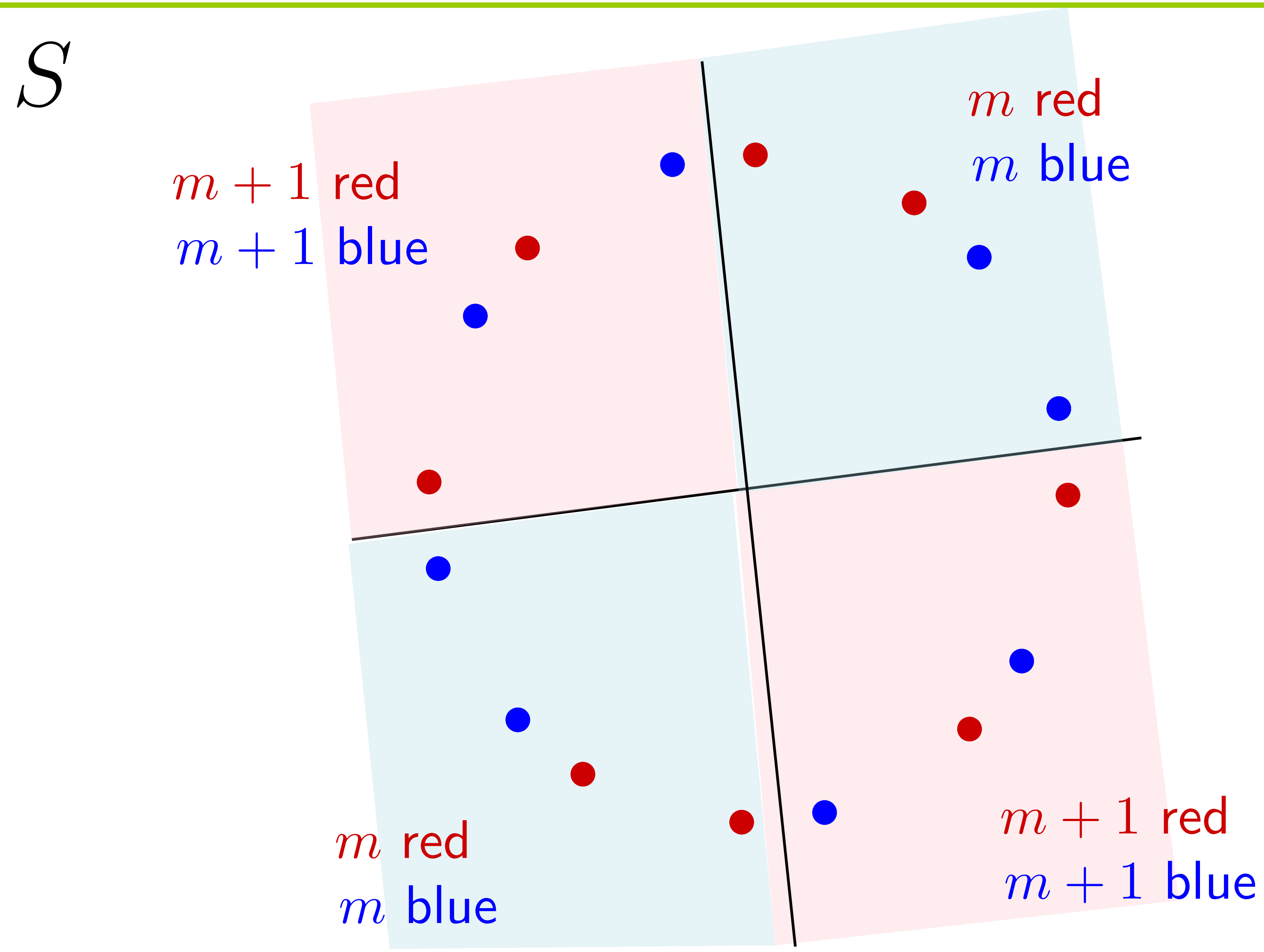
$$\text{cr}(M_P) - \text{cr}(M_Q^\vee) \geq \frac{1}{8} - c \geq 0$$

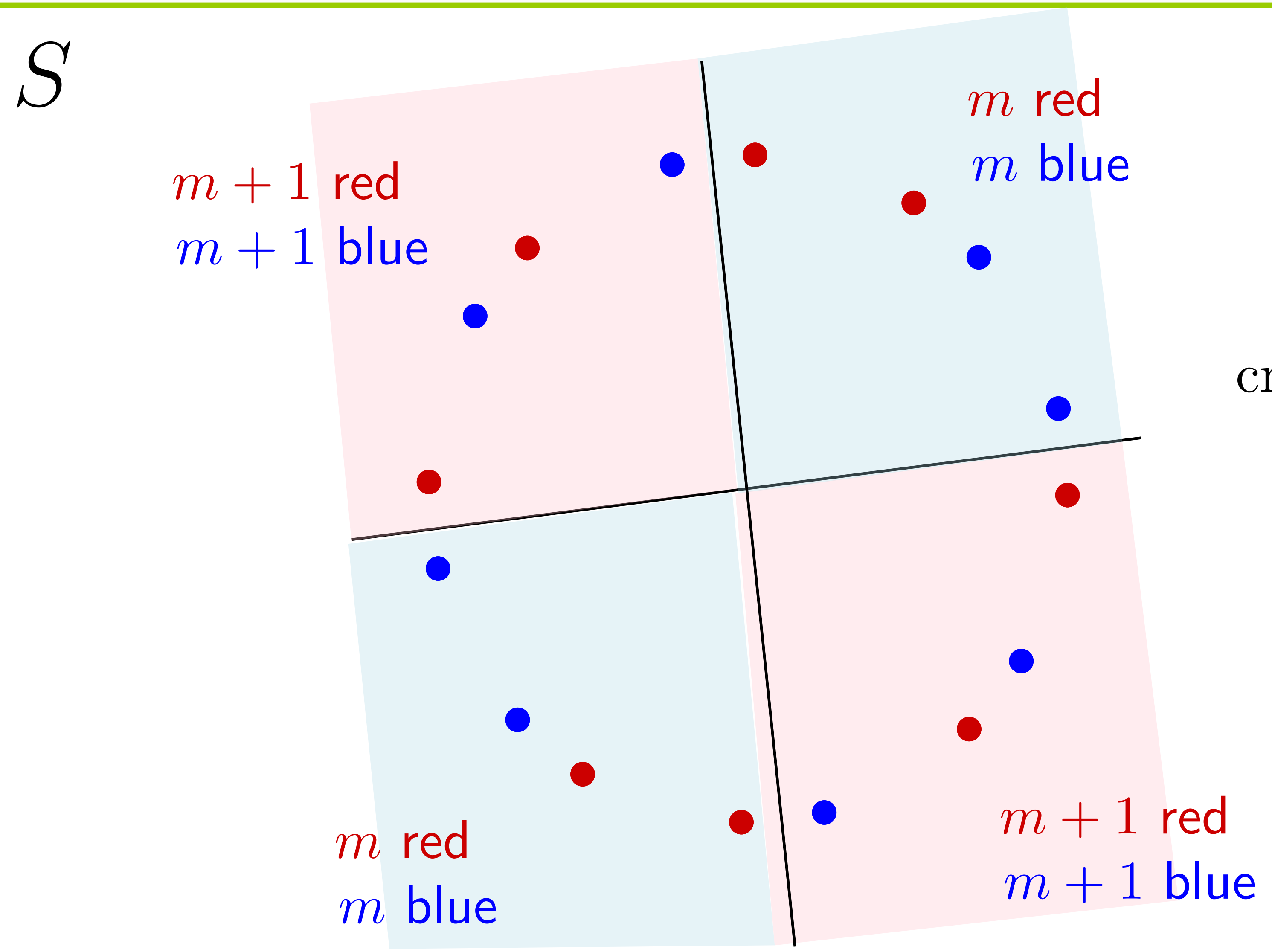
As $\text{cr}(M_P^\vee) \geq \text{cr}(M_P)$,

$$\text{cr}(M_P^\vee) - \text{cr}(M_Q^\vee) \geq 0$$

Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.



Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.



$\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.

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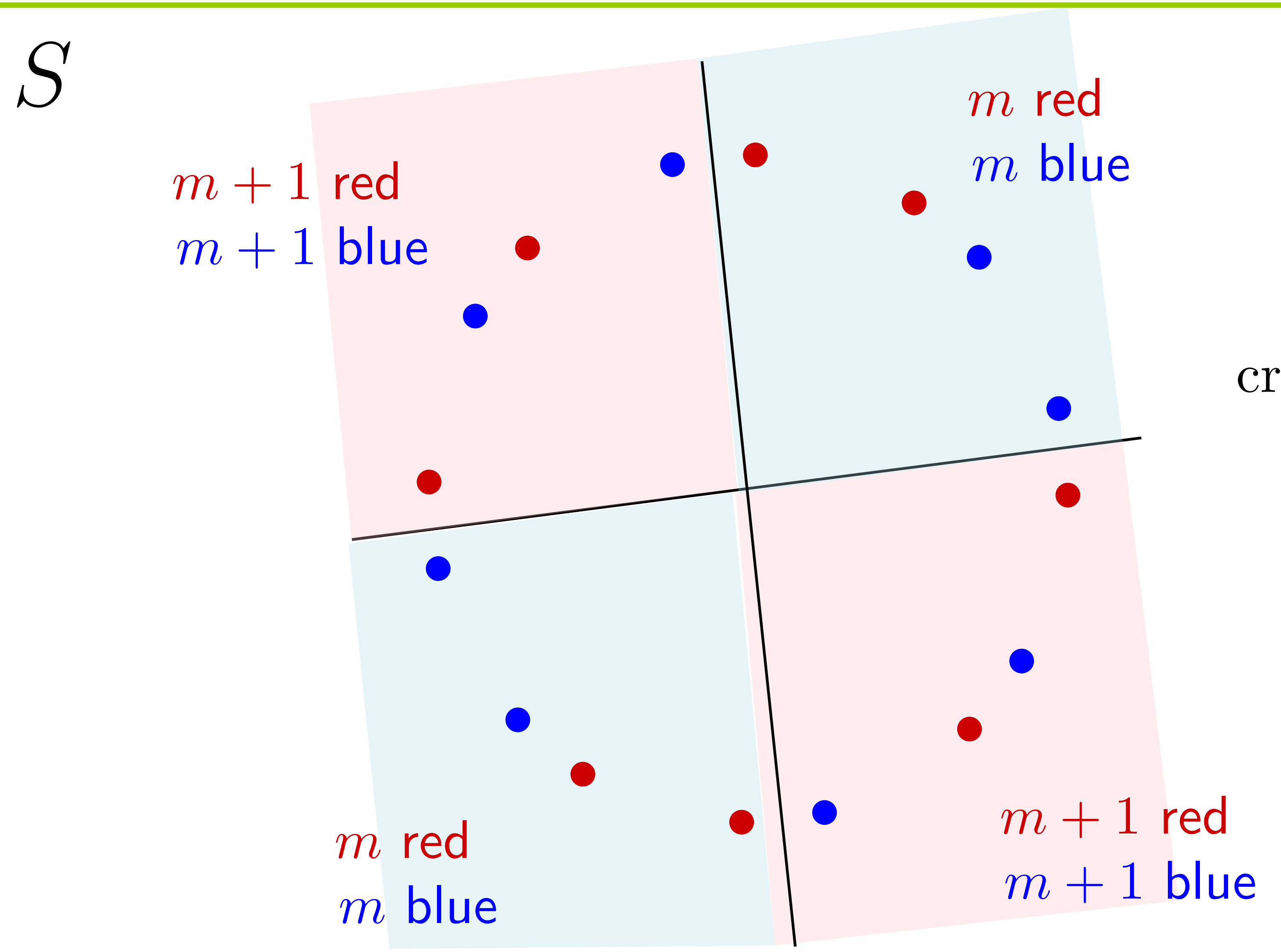
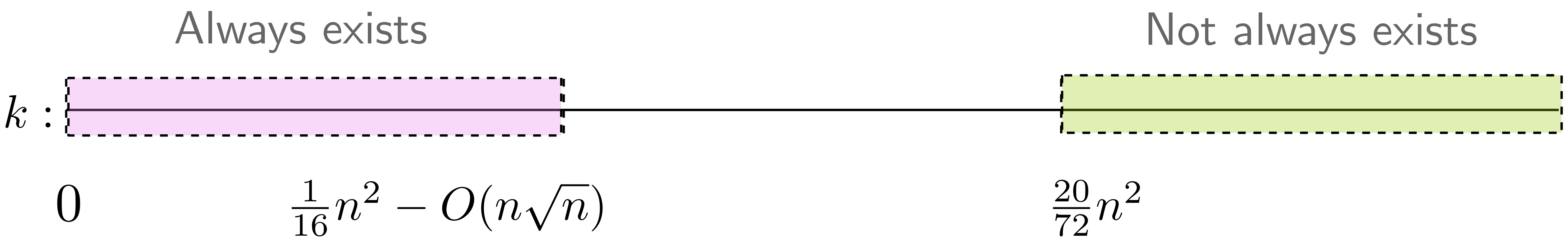


$\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.

Summary

- Every bichromatic point set in convex position admits a matching with at least $\frac{3n^2}{8} - \frac{n}{2} + c$ crossings.
- For any $k > \frac{3n^2}{8}$, there exists bichromatic convex point sets that do not admit any perfect matchings with k crossings.

Uncolored general point set



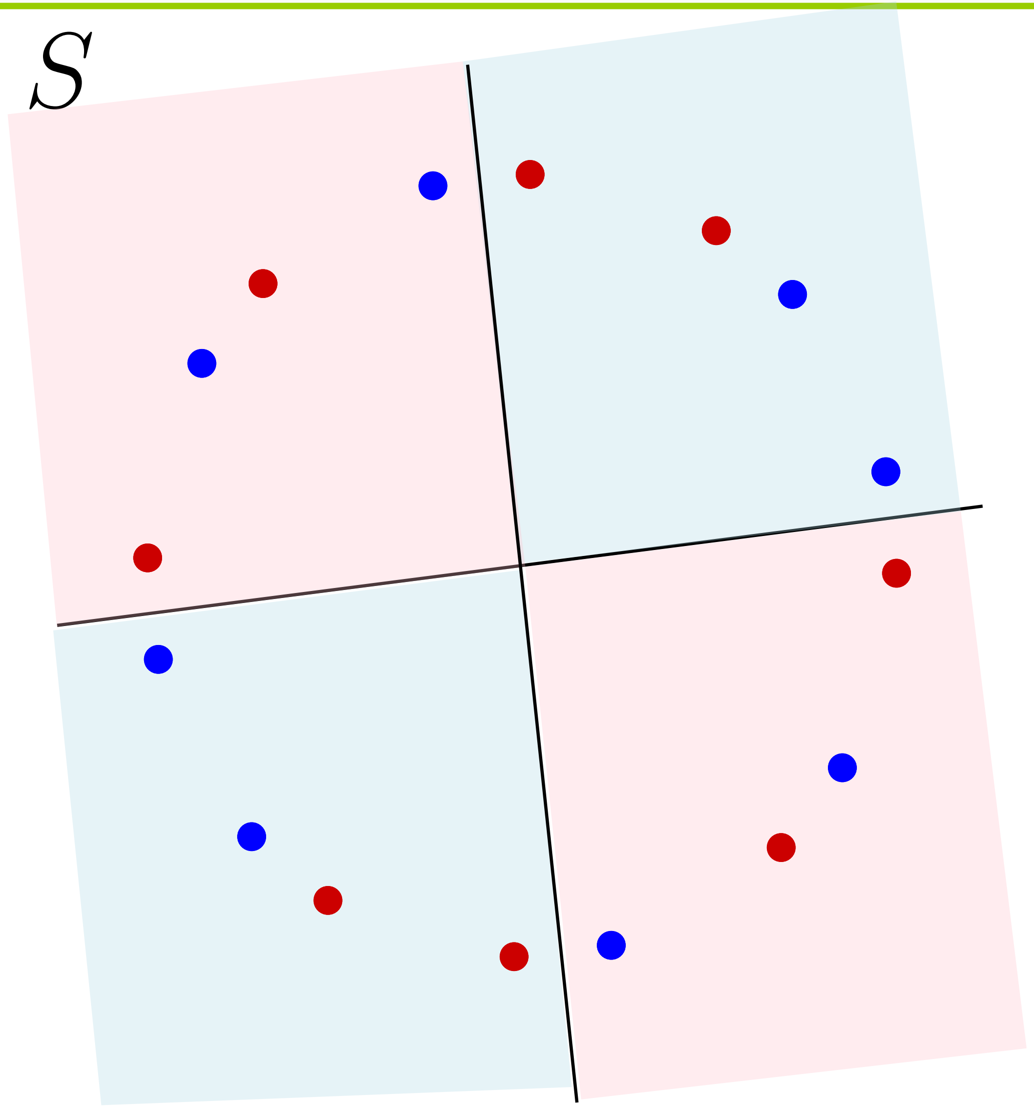
Aichholzer et. al, *Perfect matchings with crossings*, Combinatorial Algorithms, 2022.

QUESTION:

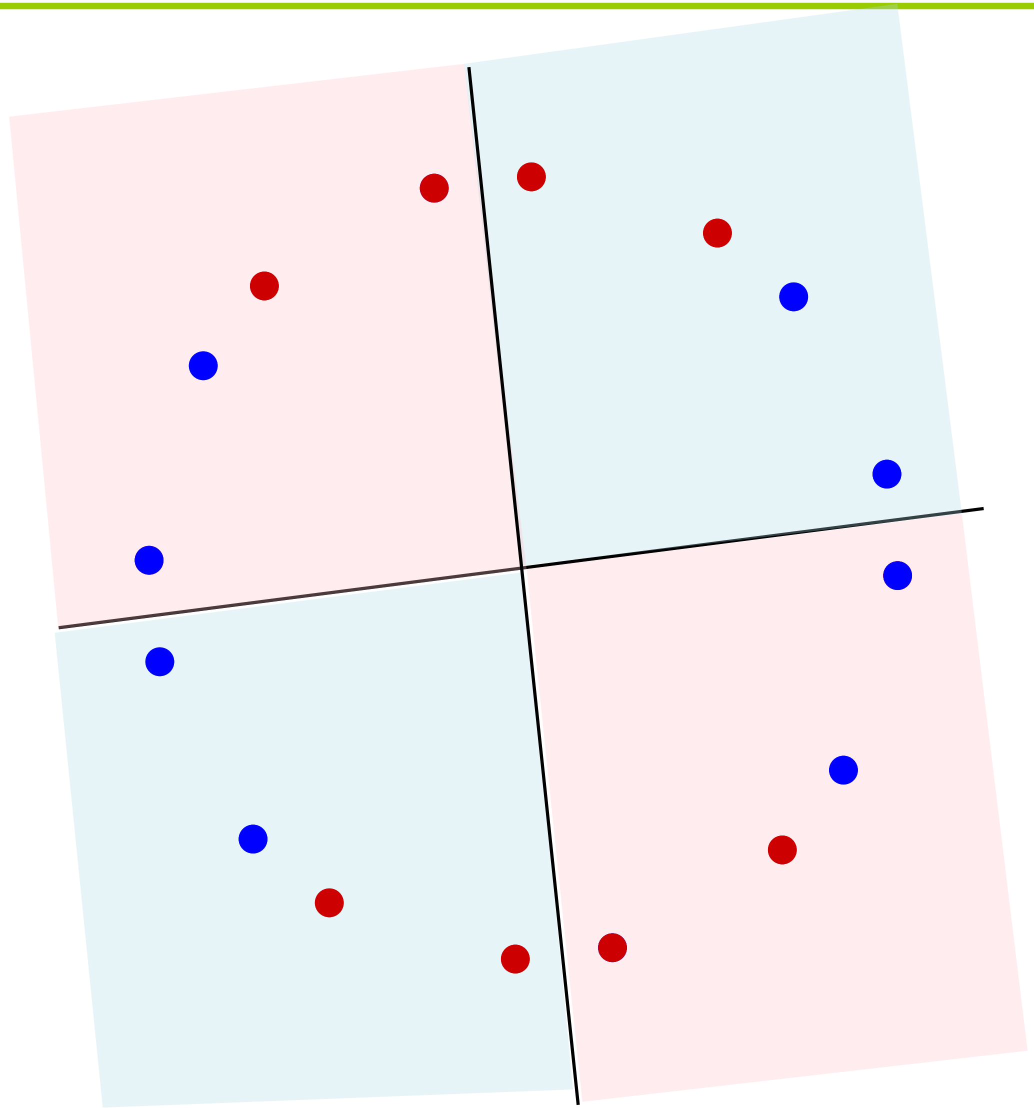
For which values of k, does there exists a bichromatic perfect matching with k crossings on all bichromatic point sets of size $2n$?

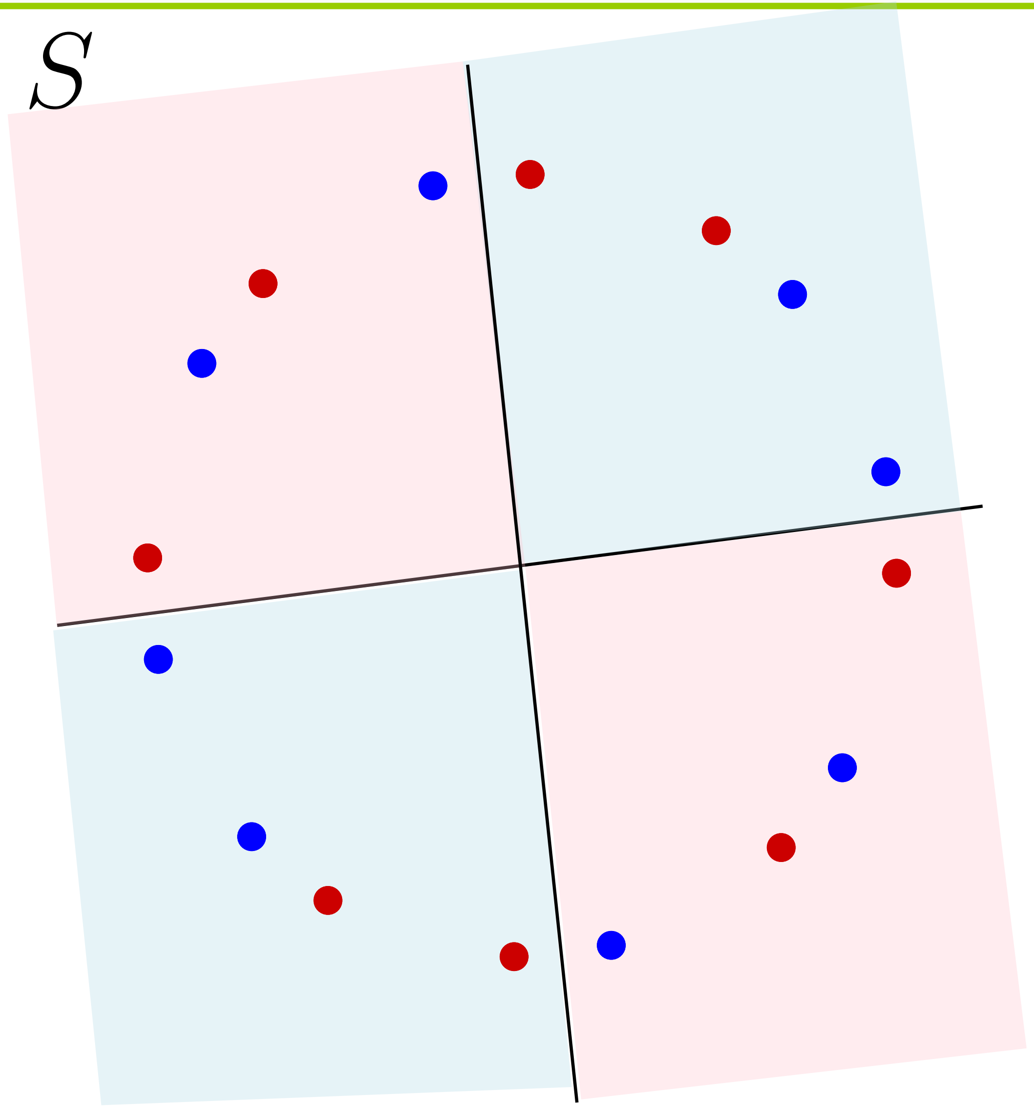
$\exists$ bichromatic point sets of size $2n$ which do not admit bichromatic perfect matchings with k crossings for $k > \frac{20n^2}{72}$.

Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.

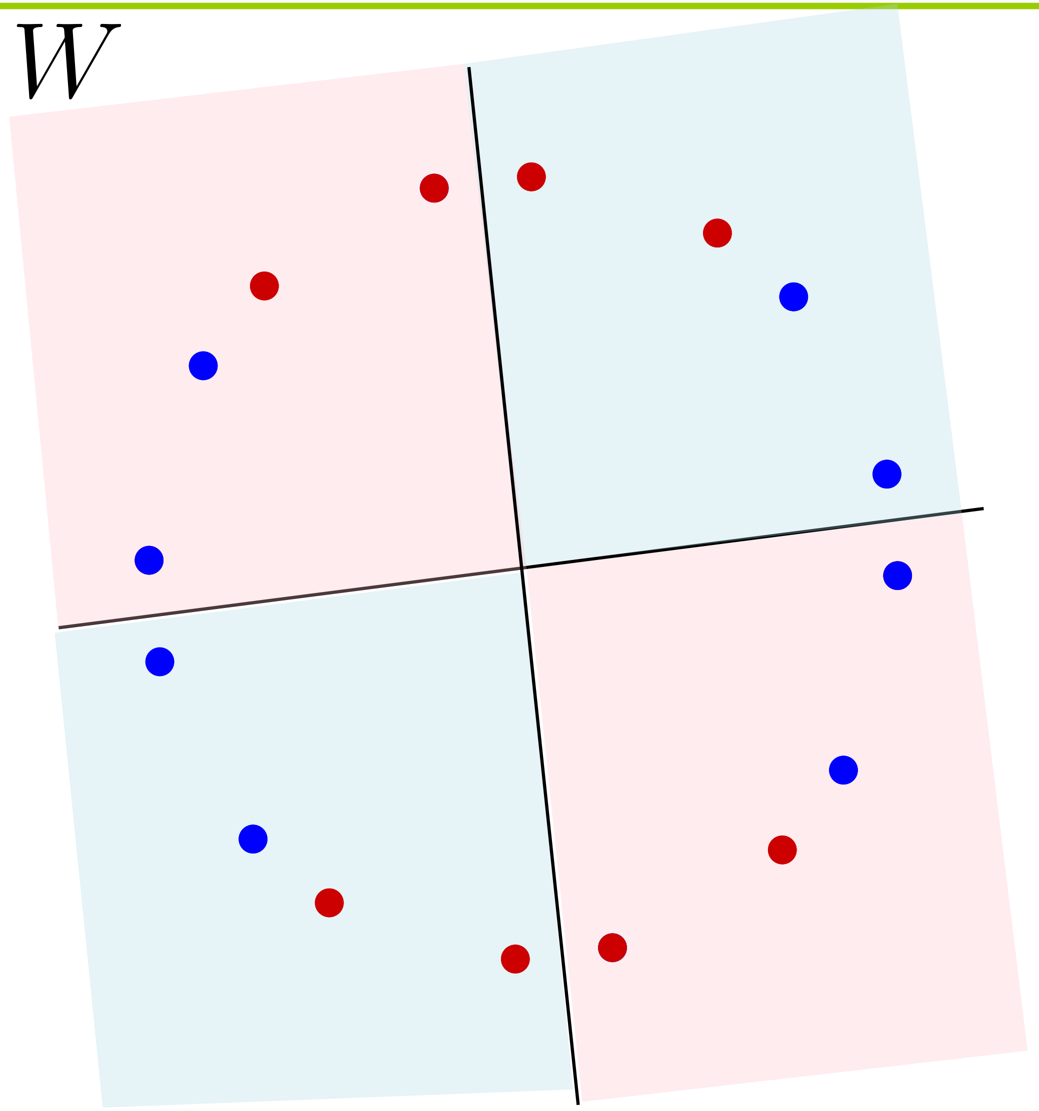


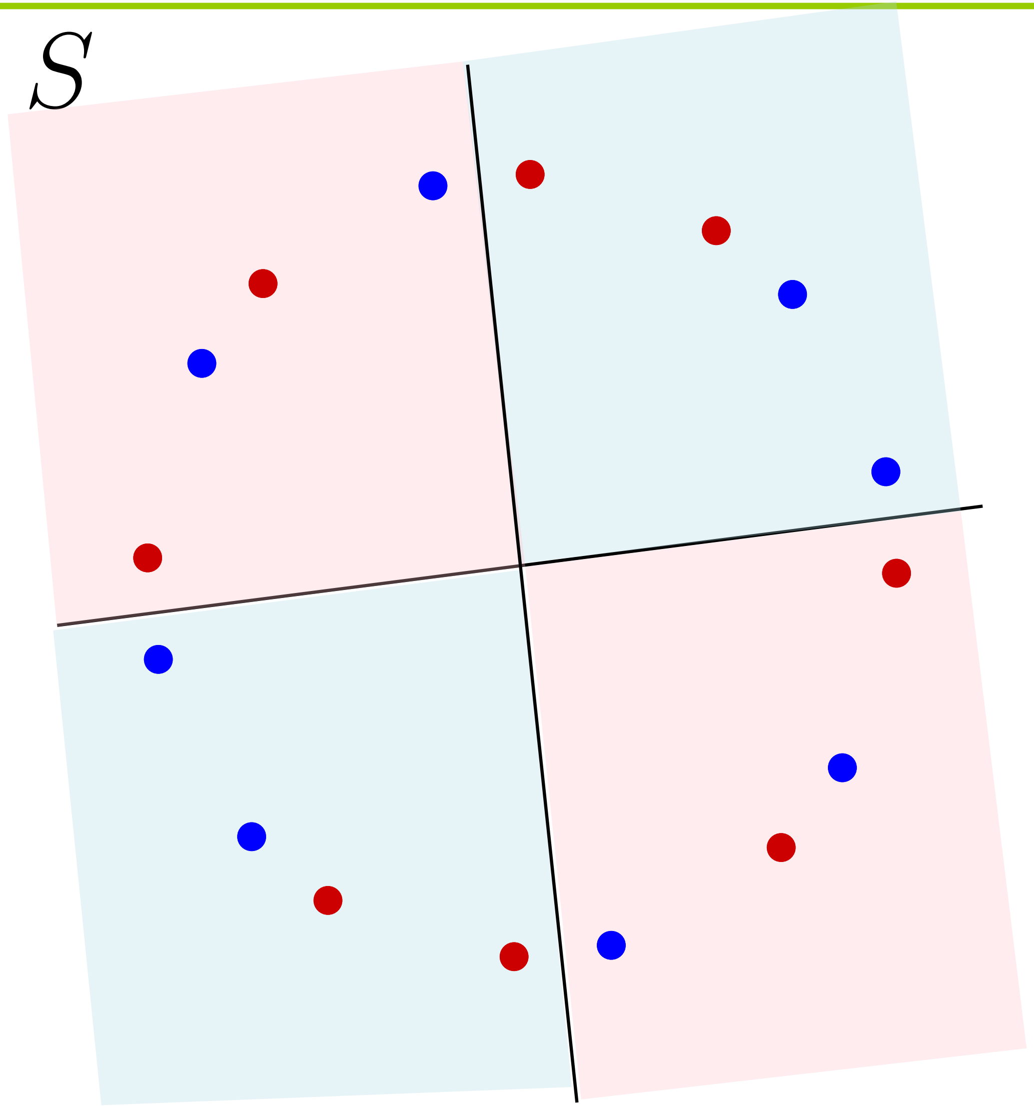
Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.





Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.





Step 2 $\mathrm{cr}(M_P^\vee) \geq \mathrm{cr}(M_Q^\vee)$.