Technische
Universität
Braunschweig

The Complexity of Recognizing Geometric Hypergraphs





Daniel Bertschinger

Nicolas El Maalouly

Linda Kleist

Tillmann Miltzow

Simon Weber

Introductory Example: Organizing a conference

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar
- Straight-line
- Crossings
- Beyond planarity

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line
- Crossings
- Beyond planarity

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings
- Beyond planarity

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

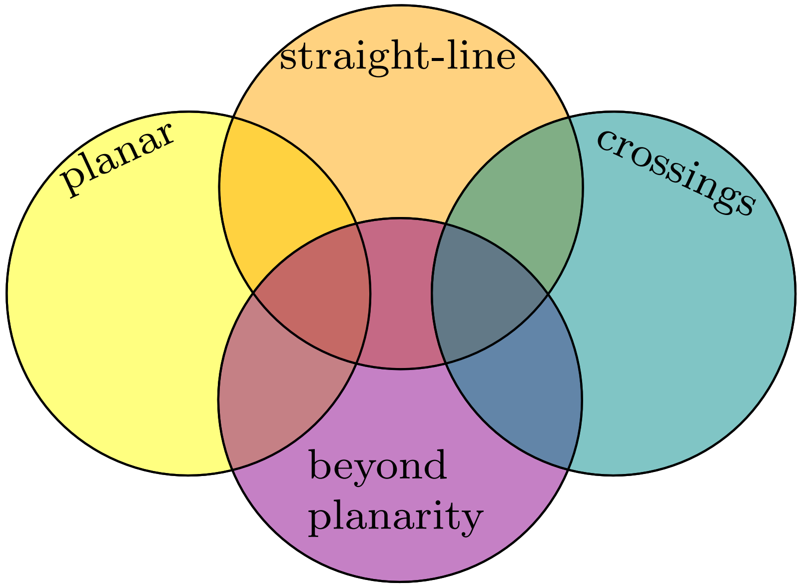
- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$

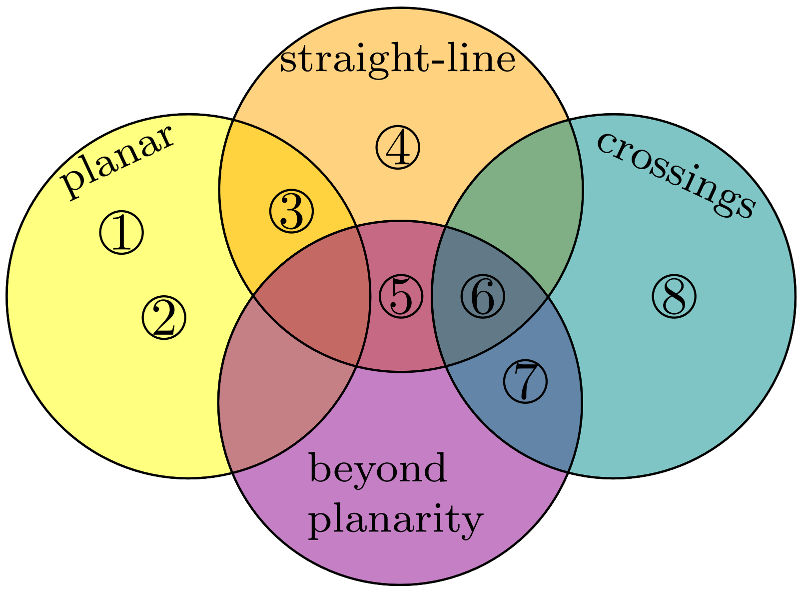


Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$

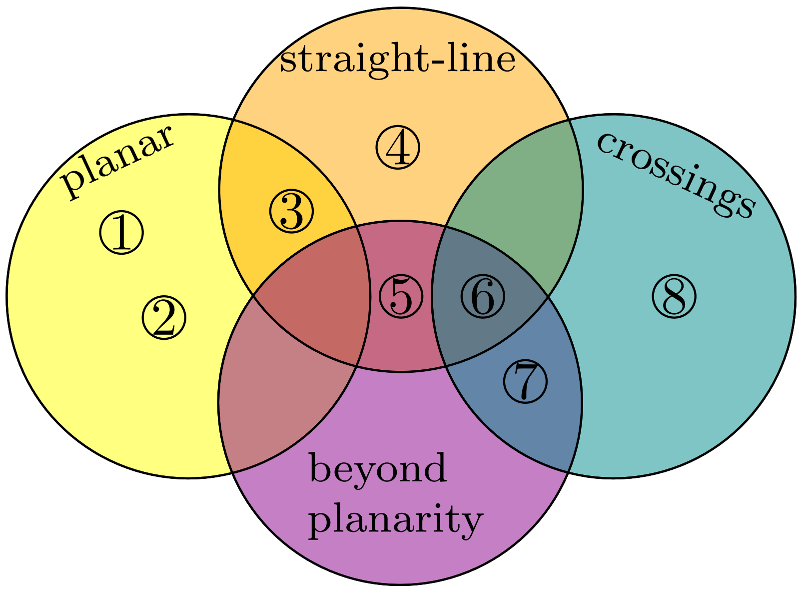


Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,...,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



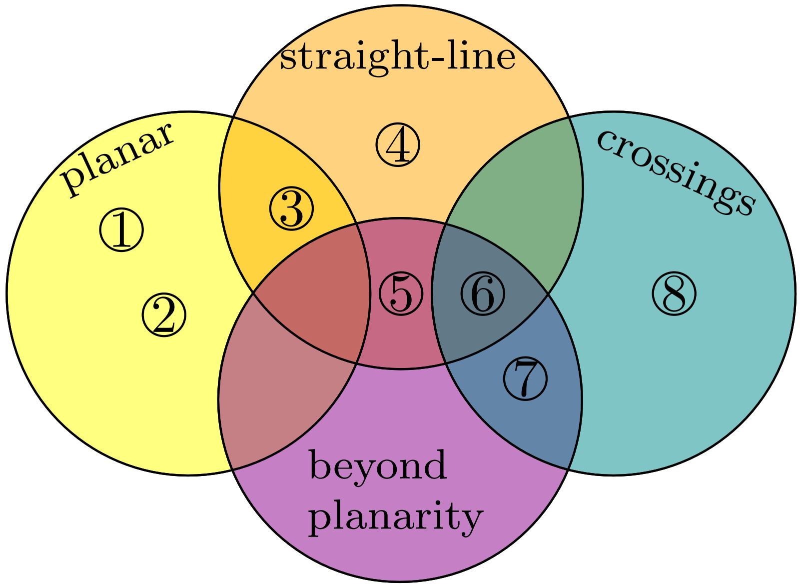
hypergraph $H = (V, E)$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

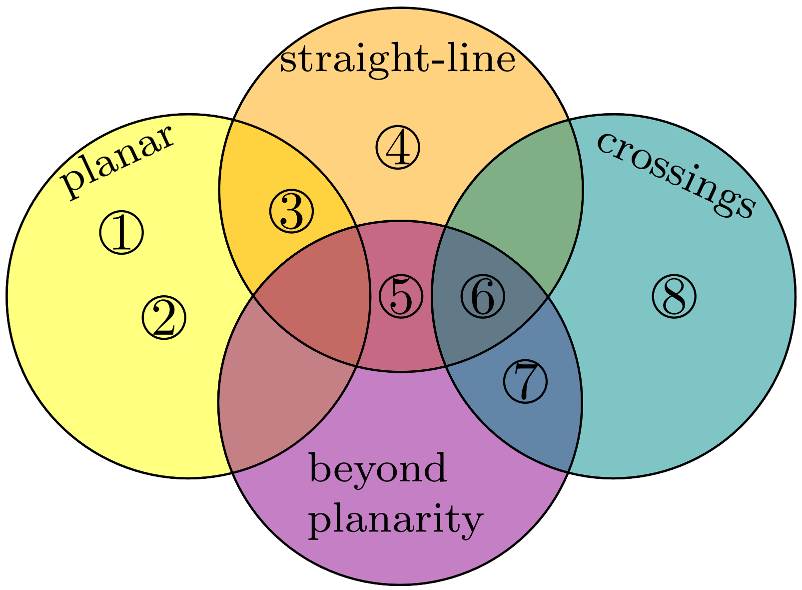
geometric representation

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,...,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

geometric representation

vertex $v \in V \rightarrow$ point $p_v \in \mathbb{R}^d$

hyperedge $e \in E \rightarrow$ set $s_e \subset \mathbb{R}^d$ s.t.

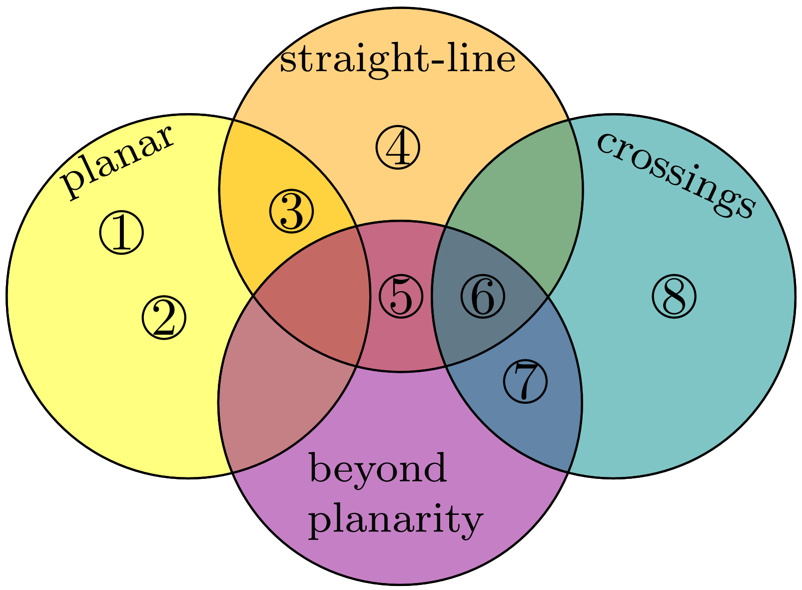
$$p_v \in s_e \iff v \in e$$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

geometric representation

family $\mathscr{F}$ of sets in $\mathbb{R}^d$

vertex $v \in V$ $\rightarrow$ point $p_v \in \mathbb{R}^d$

hyperedge $e \in E \rightarrow$ set $s_e \subset \mathbb{R}^d$ s.t.

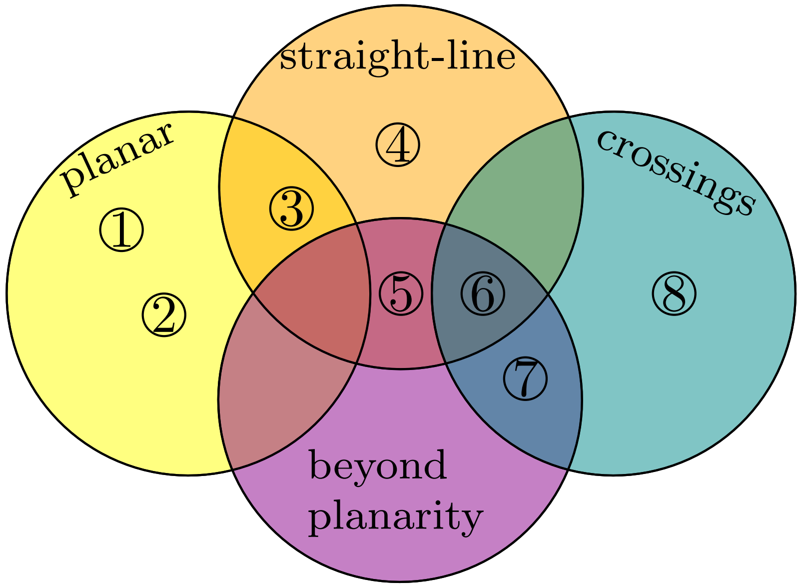
$$p_v \in s_e \iff v \in e$$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,...,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

$\mathscr{F}$ - geometric representation

family $\mathscr{F}$ of sets in $\mathbb{R}^d$

vertex $v \in V$ $\rightarrow$ point $p_v \in \mathbb{R}^d$

hyperedge $e \in E \rightarrow$ set $s_e \subset \mathbb{R}^d$ s.t.

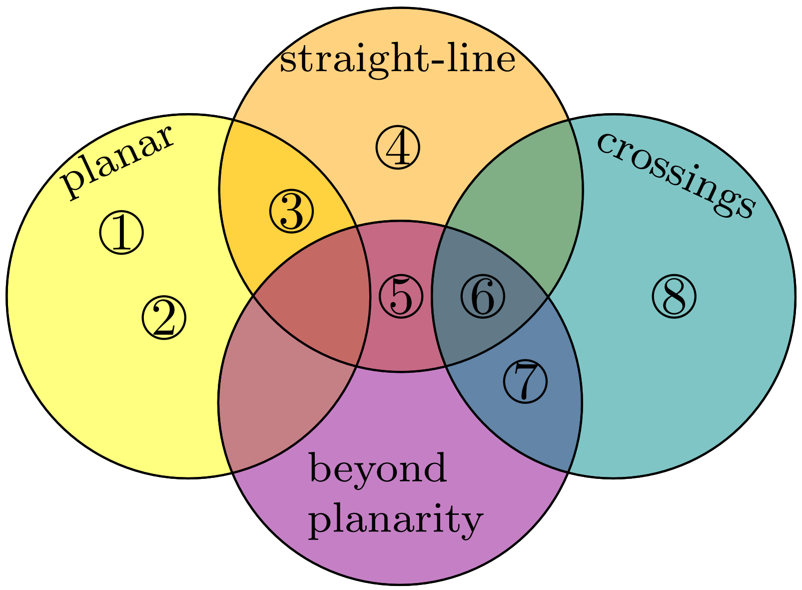
$$p_v \in s_e \iff v \in e$$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

$\mathscr{F}$-geometric representation

family $\mathscr{F}$ of sets in $\mathbb{R}^d$

vertex $v \in V$ $\rightarrow$ point $p_v \in \mathbb{R}^d$

hyperedge $e \in E \rightarrow$ set $s_e \in \mathscr{F}$ s.t.

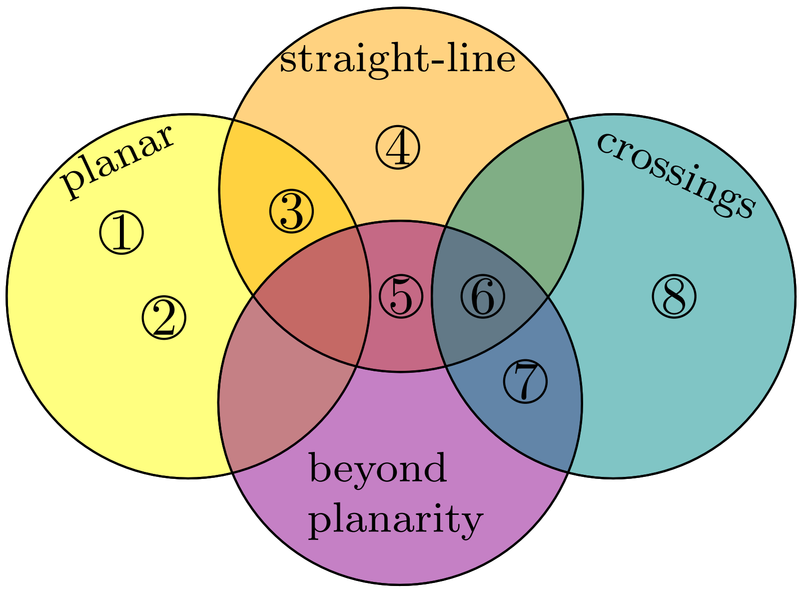
$p_v \in s_e \iff v \in e$

Introductory Example: Organizing a conference

Accepted papers $= \{1,2,3,\ldots,n\}$

Categories

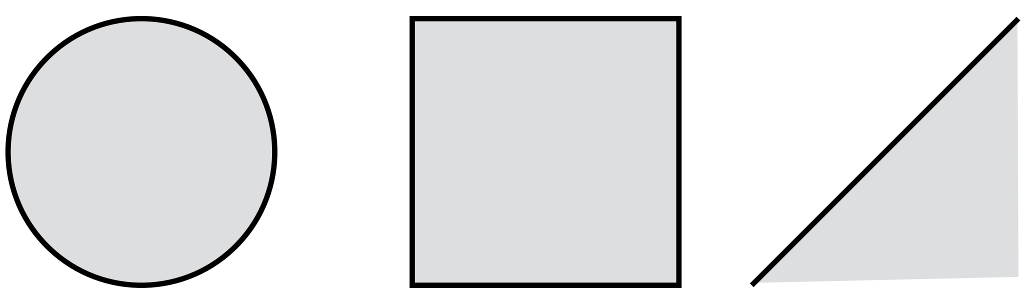
- Planar $\{1,2,\ldots\}$
- Straight-line $\{3,4,5,6,\ldots\}$
- Crossings $\{6,7,8,\ldots\}$
- Beyond planarity $\{5,6,7,\ldots\}$



hypergraph $H = (V, E)$

$\mathcal{F}$-geometric representation

family $\mathcal{F}$ of sets in $\mathbb{R}^d$



vertex $v \in V$ $\rightarrow$ point $p_v \in \mathbb{R}^d$

hyperedge $e \in E \rightarrow$ set $s_e \in \mathcal{F}$ s.t.

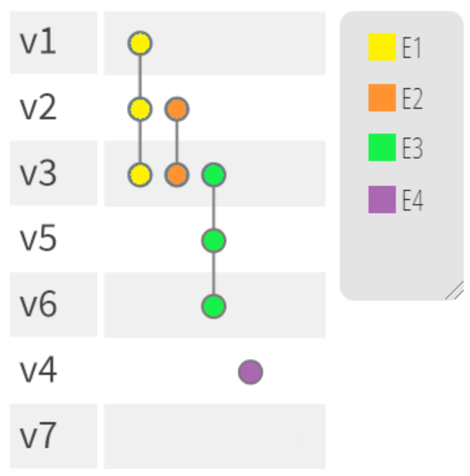
$$p_v \in s_e \iff v \in e$$

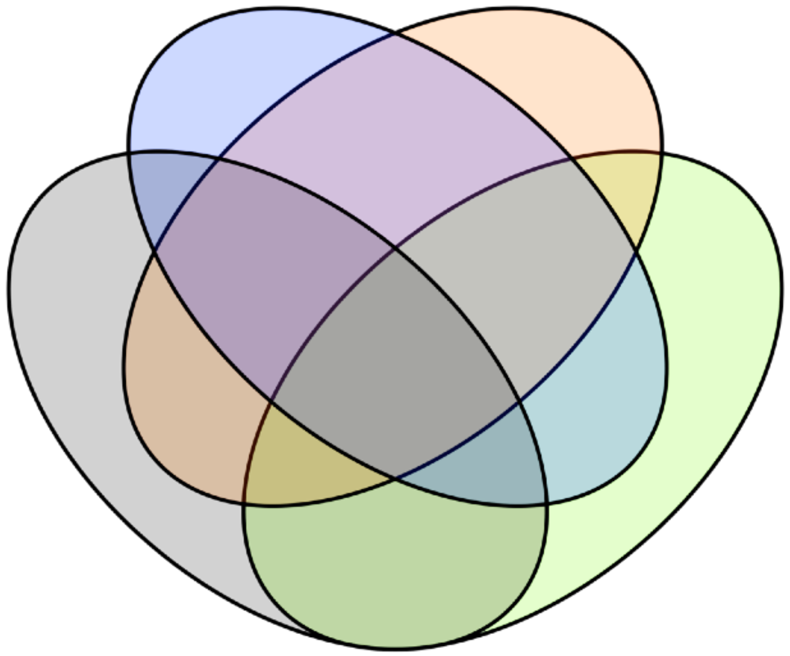
How to draw a hypergraph?

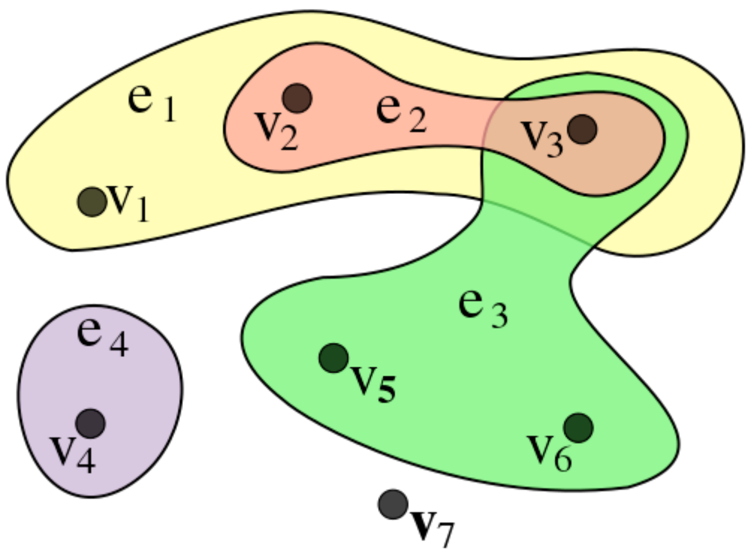
How to draw a hypergraph?

Wikipedia suggests...

Wikipedia suggests...



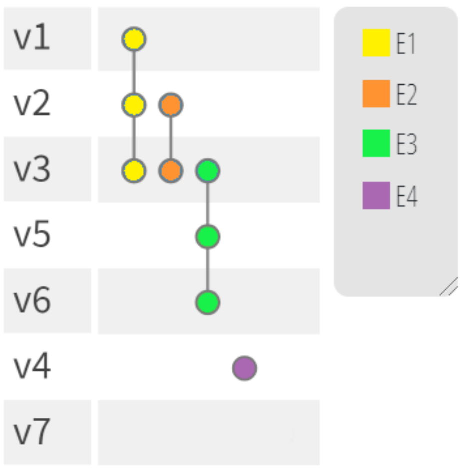




... and
many more

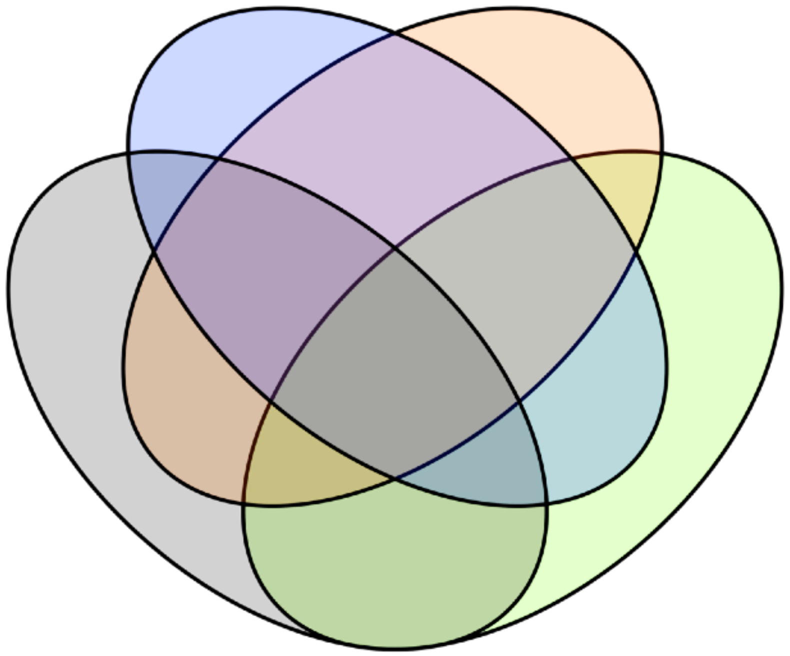
How to draw a hypergraph?

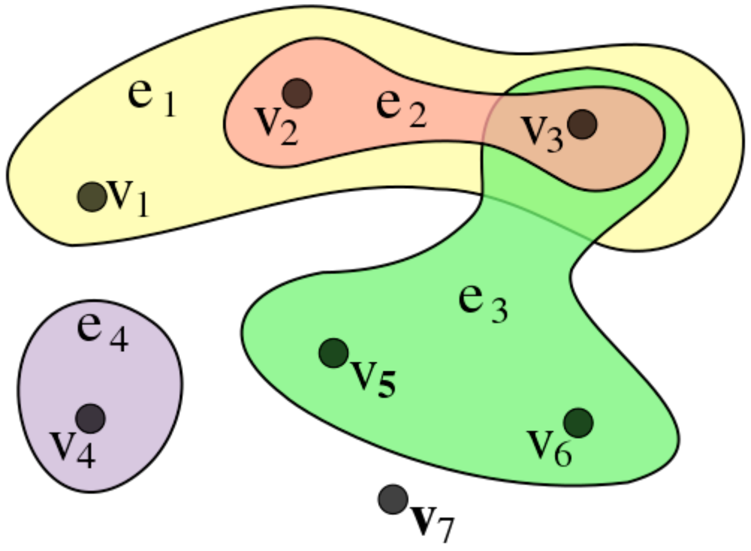
Wikipedia suggests...



PAOH

Parallel
Aggregated
Ordered
Hypergraph

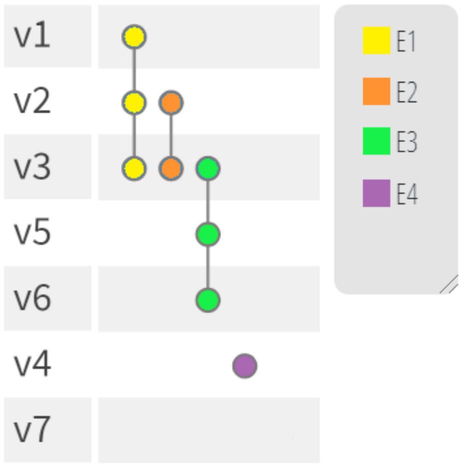




... and many more

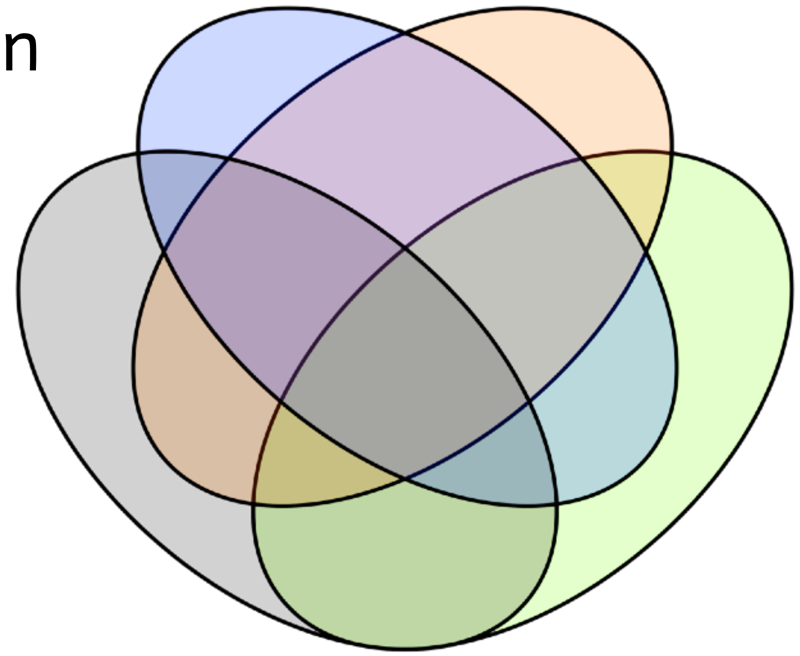
How to draw a hypergraph?

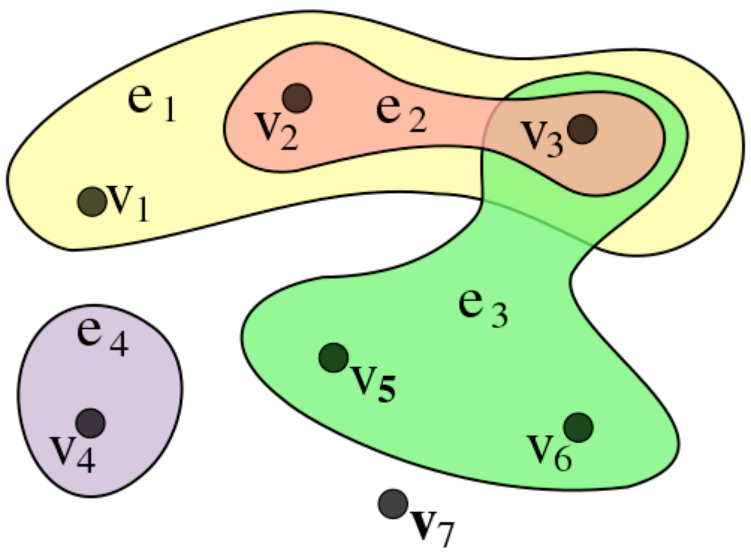
Wikipedia suggests...



PAOH
Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings

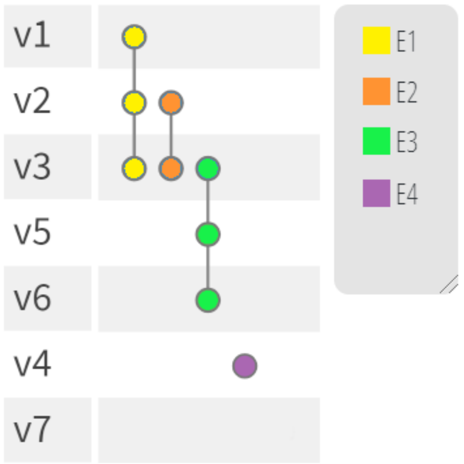




... and many more

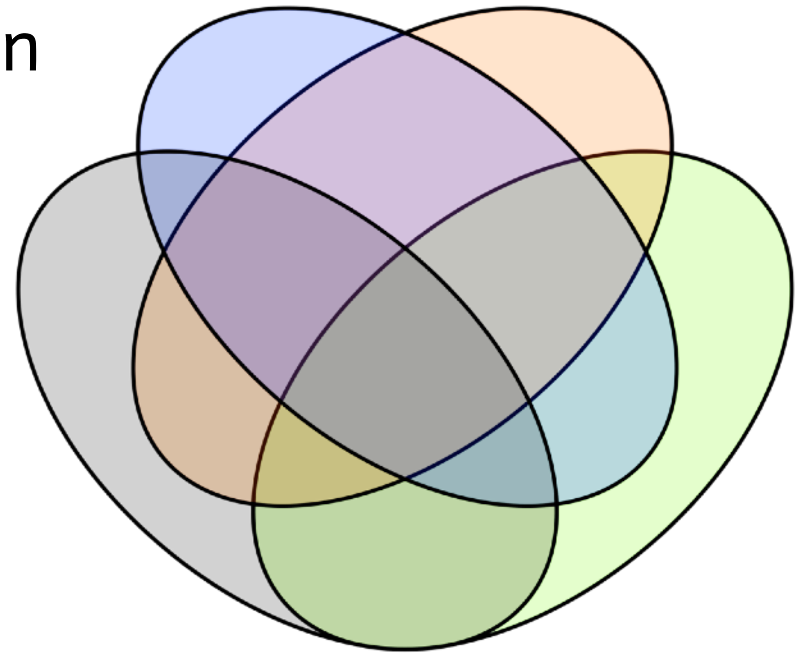
How to draw a hypergraph?

Wikipedia suggests…

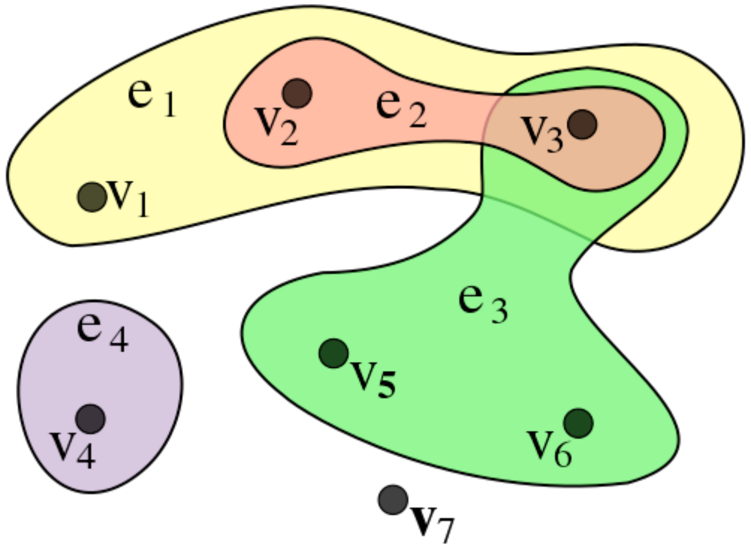


PAOH
Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings



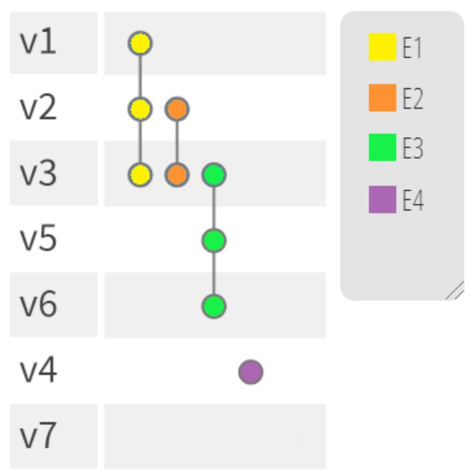
geometric representation (subset standard)



… and many more

How to draw a hypergraph?

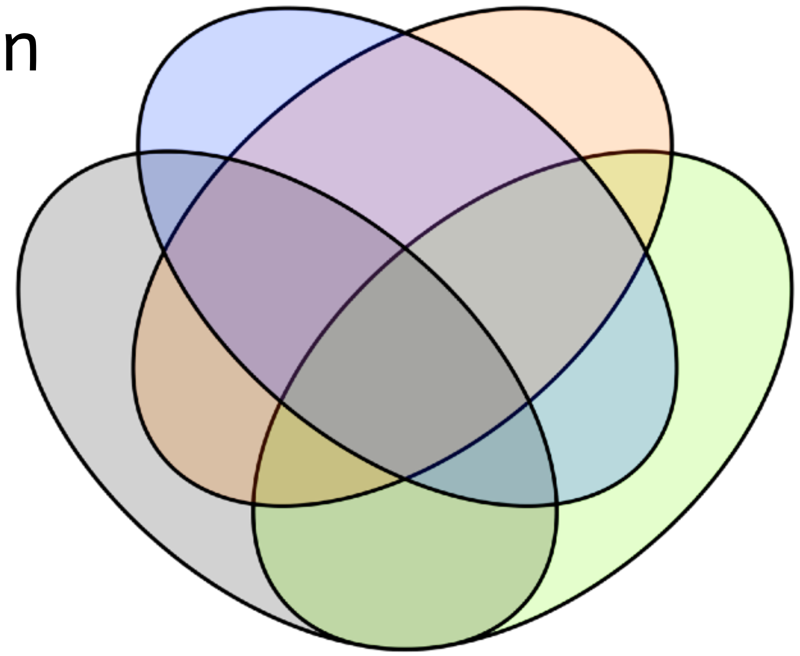
Wikipedia suggests…



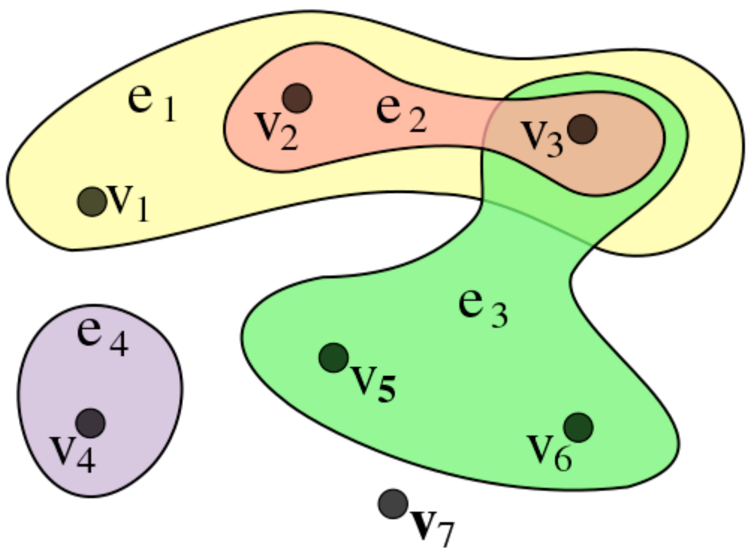
PAOH

Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings



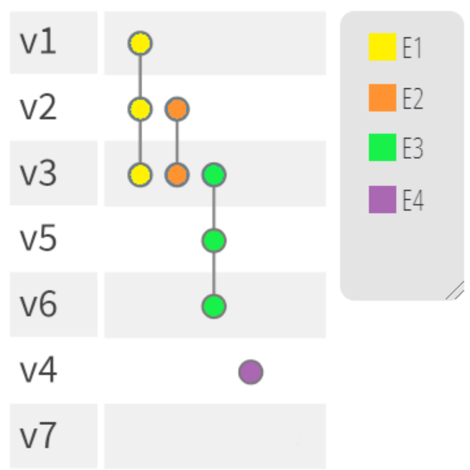
geometric representation (subset standard)



… and many more

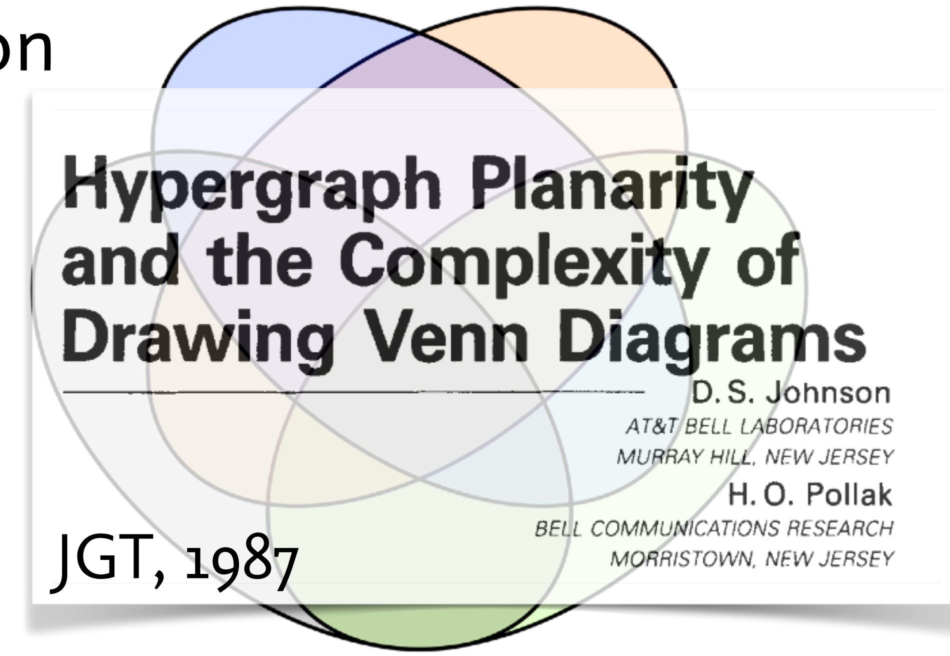
How to draw a hypergraph?

Wikipedia suggests...



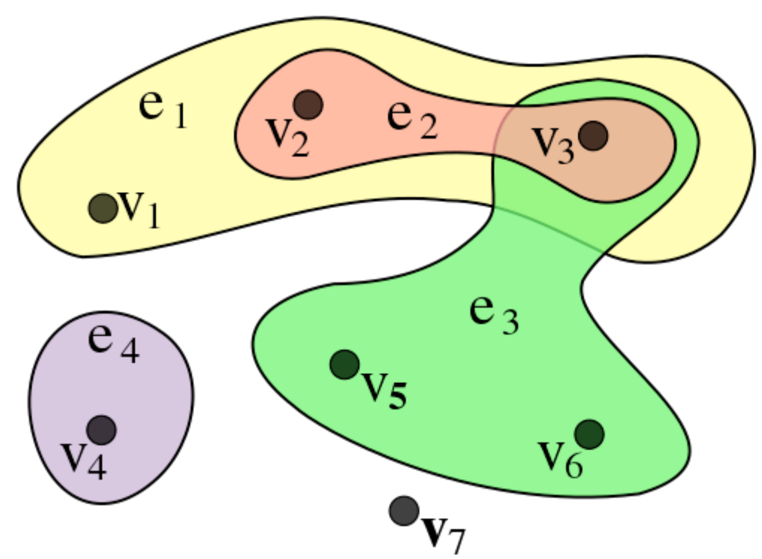
PAOH
Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings


Hypergraph Planarity and the Complexity of Drawing Venn Diagrams
D. S. Johnson
AT&T BELL LABORATORIES
MURRAY HILL, NEW JERSEY
H. O. Pollak
BELL COMMUNICATIONS RESEARCH
MORRISTOWN, NEW JERSEY

JGT, 1987

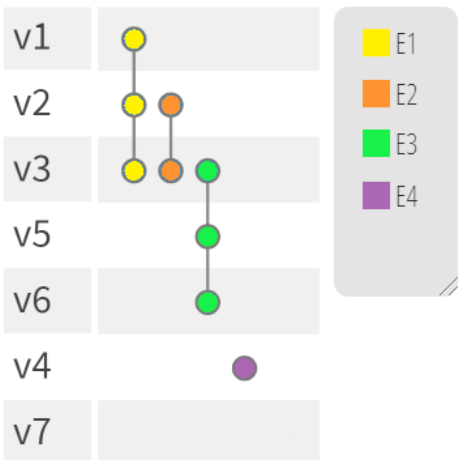
geometric representation (subset standard)



... and many more

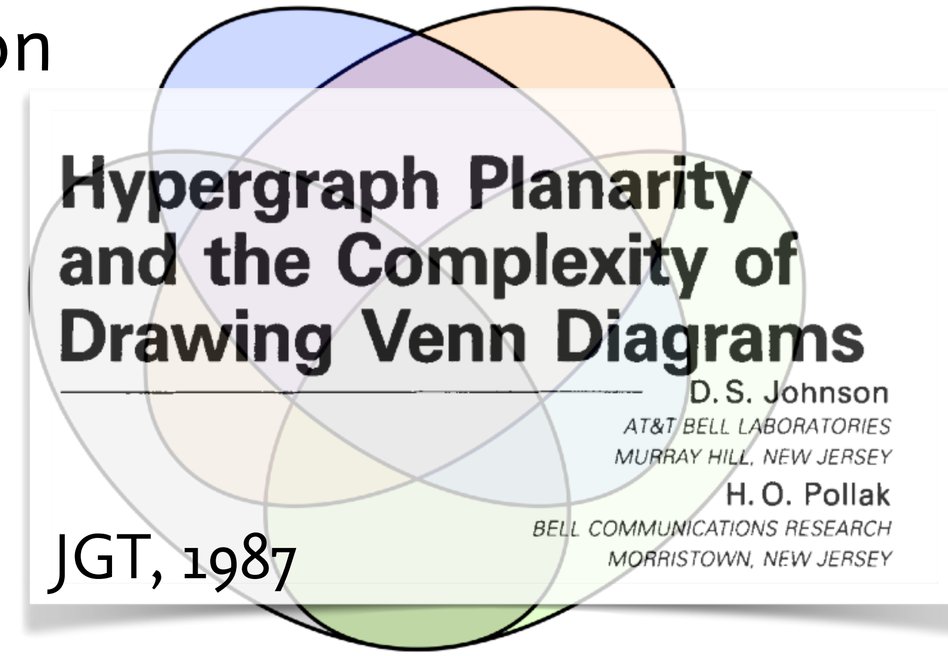
How to draw a hypergraph?

Wikipedia suggests...



PAOH
Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings

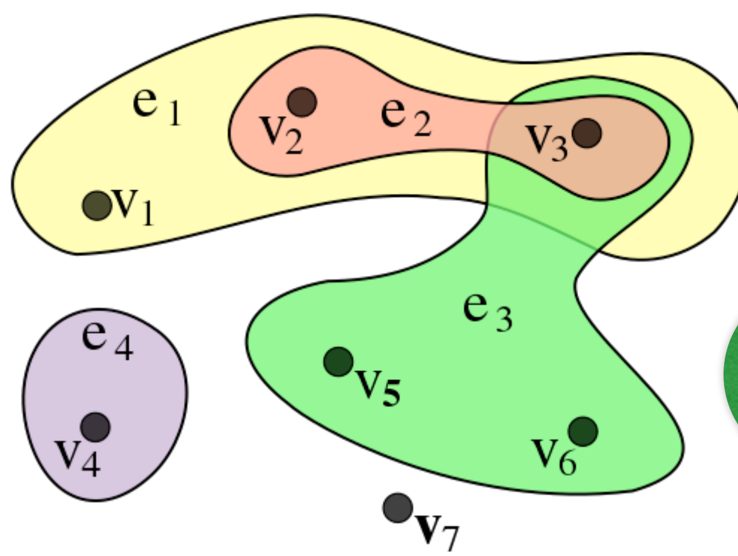

Hypergraph Planarity and the Complexity of Drawing Venn Diagrams

D. S. Johnson
AT&T BELL LABORATORIES
MURRAY HILL, NEW JERSEY

H. O. Pollak
BELL COMMUNICATIONS RESEARCH
MORRISTOWN, NEW JERSEY

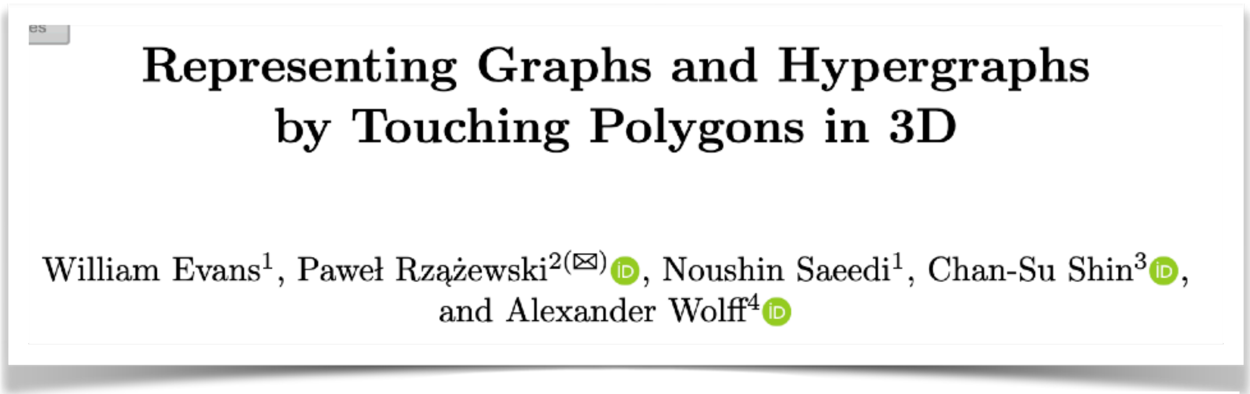
JGT, 1987

geometric representation (subset standard)

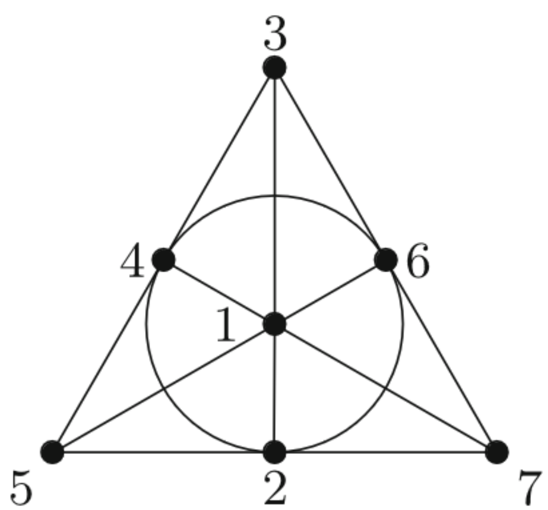


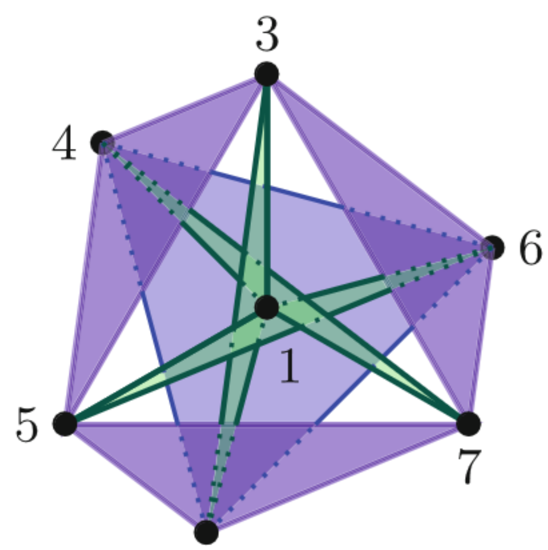
... and many more

How to draw a hypergraph?


Representing Graphs and Hypergraphs by Touching Polygons in 3D

William Evans[1], Paweł Rzążewski[2(✉)], Noushin Saeedi[1], Chan-Su Shin[3], and Alexander Wolff[4]



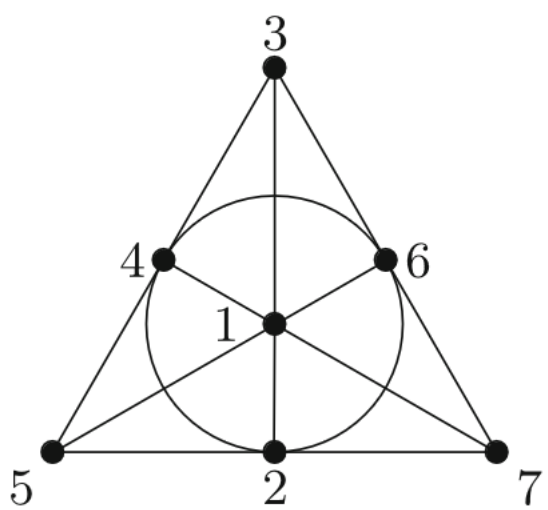


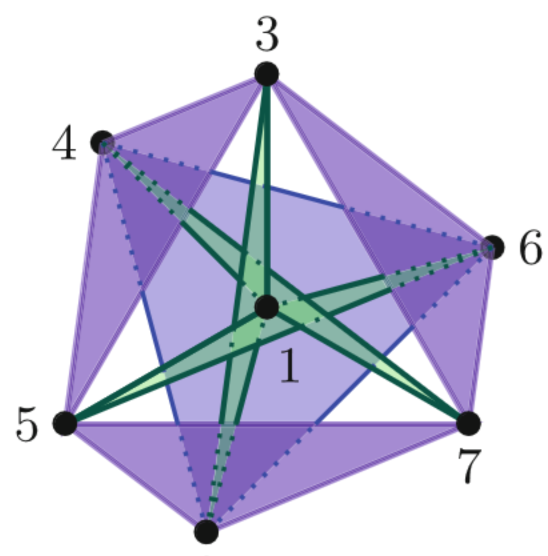
touching polygons

How to draw a hypergraph?

Representing Graphs and Hypergraphs by Touching Polygons in 3D

William Evans[1], Paweł Rzążewski[2(✉)], Noushin Saeedi[1], Chan-Su Shin[3], and Alexander Wolff[4]





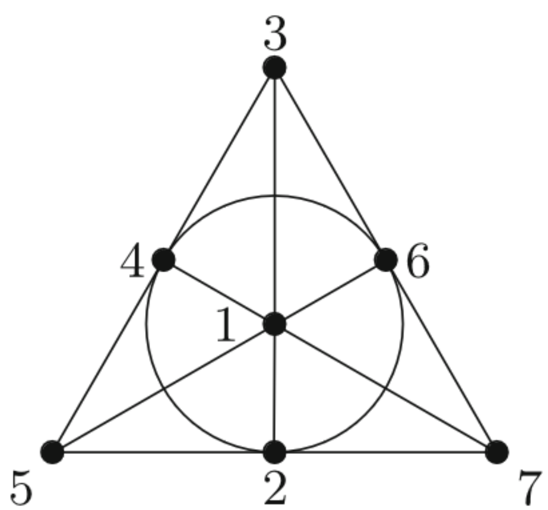
touching polygons

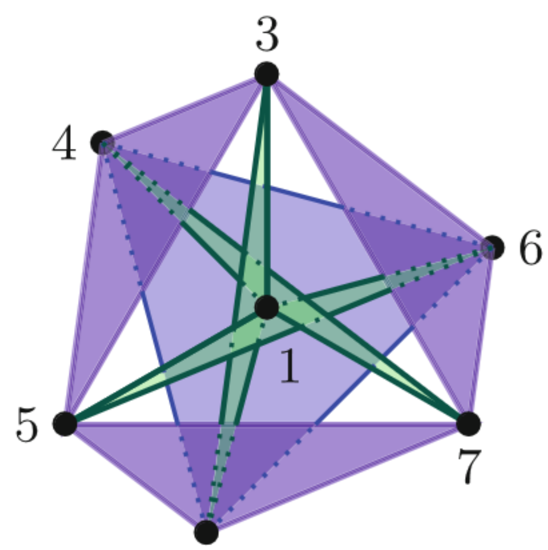
simplicial complex

How to draw a hypergraph?

Representing Graphs and Hypergraphs by Touching Polygons in 3D

William Evans[1], Paweł Rzążewski[2(✉)], Noushin Saeedi[1], Chan-Su Shin[3], and Alexander Wolff[4]

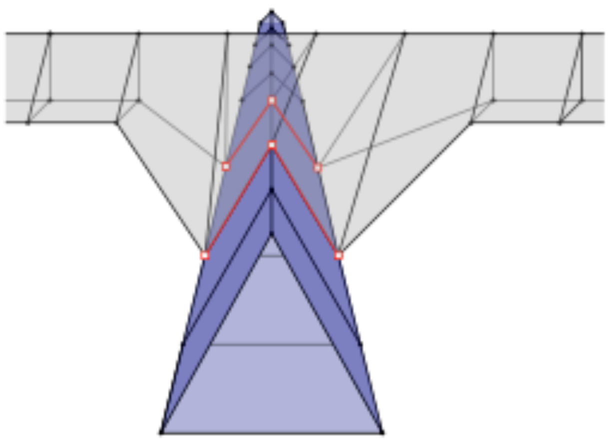




touching polygons

Geometric Embeddability of Complexes Is ∃ℝ-Complete

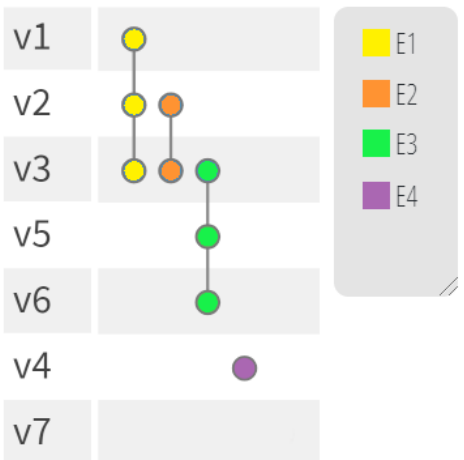
Mikkel Abrahamsen ✉
University of Copenhagen, Denmark
Linda Kleist ✉
Technische Universität Braunschweig, Germany
Tillmann Miltzow ✉
Utrecht University, The Netherlands



simplicial complex

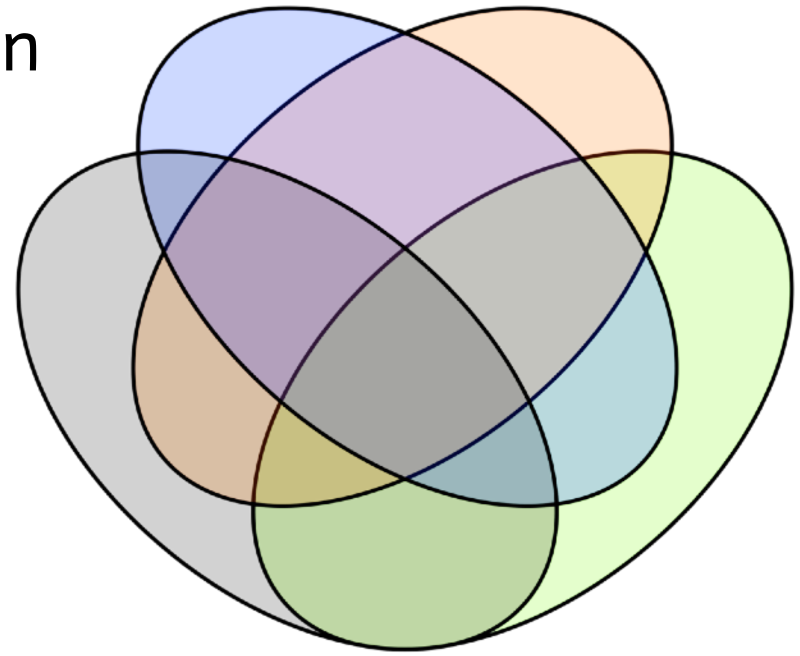
How to draw a hypergraph?

Wikipedia suggests...

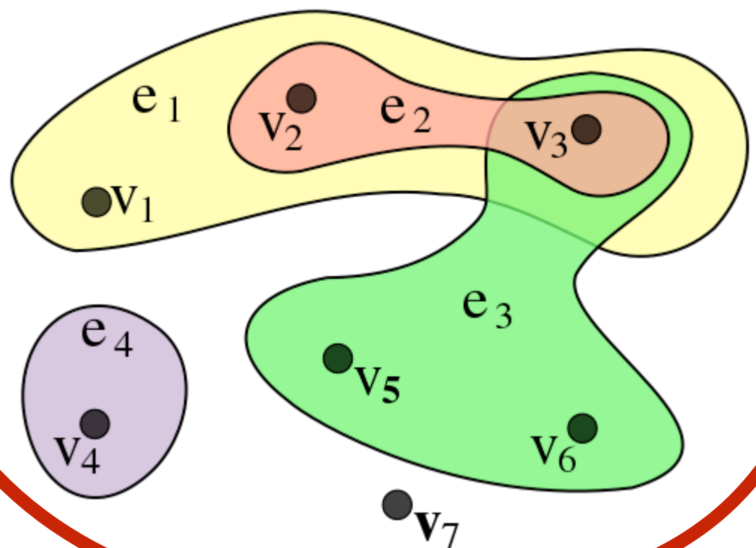


PAOH
Parallel
Aggregated
Ordered
Hypergraph

subdivision drawings

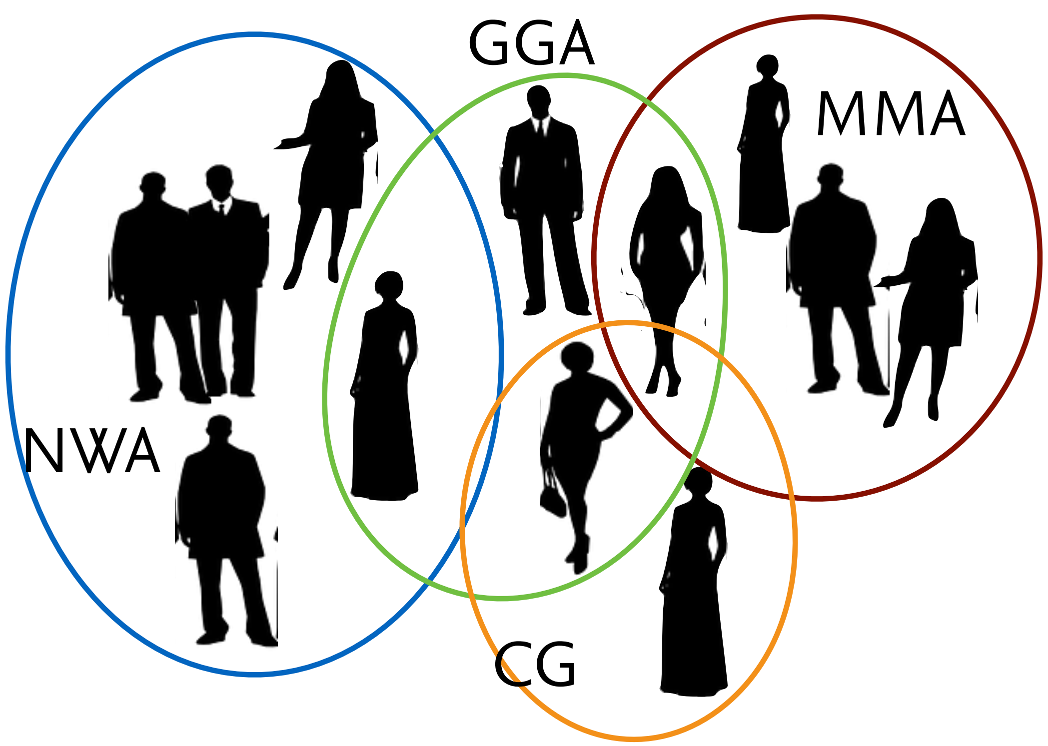


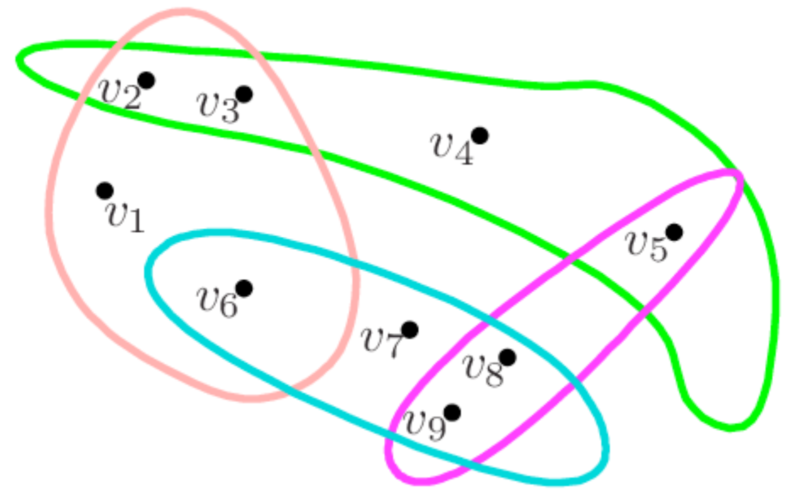
geometric representation (subset standard)

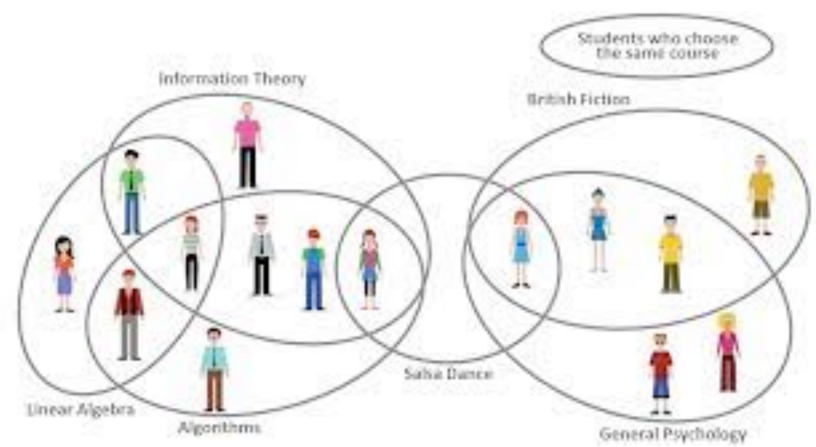


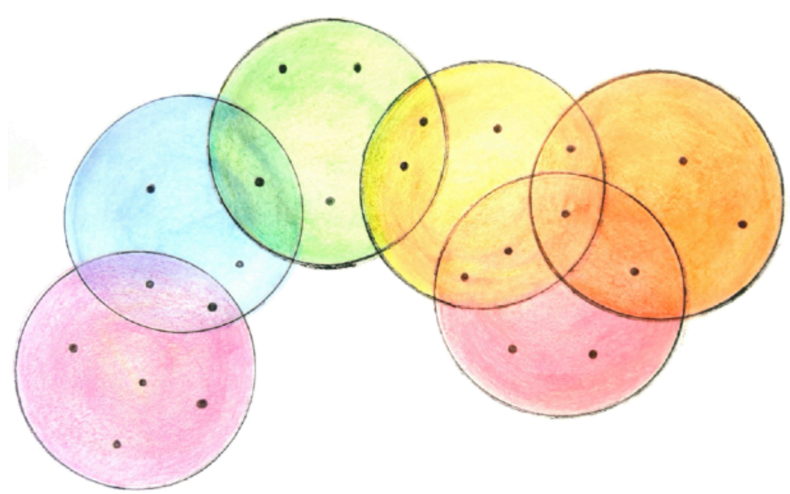
... and many more

How to draw a hypergraph?

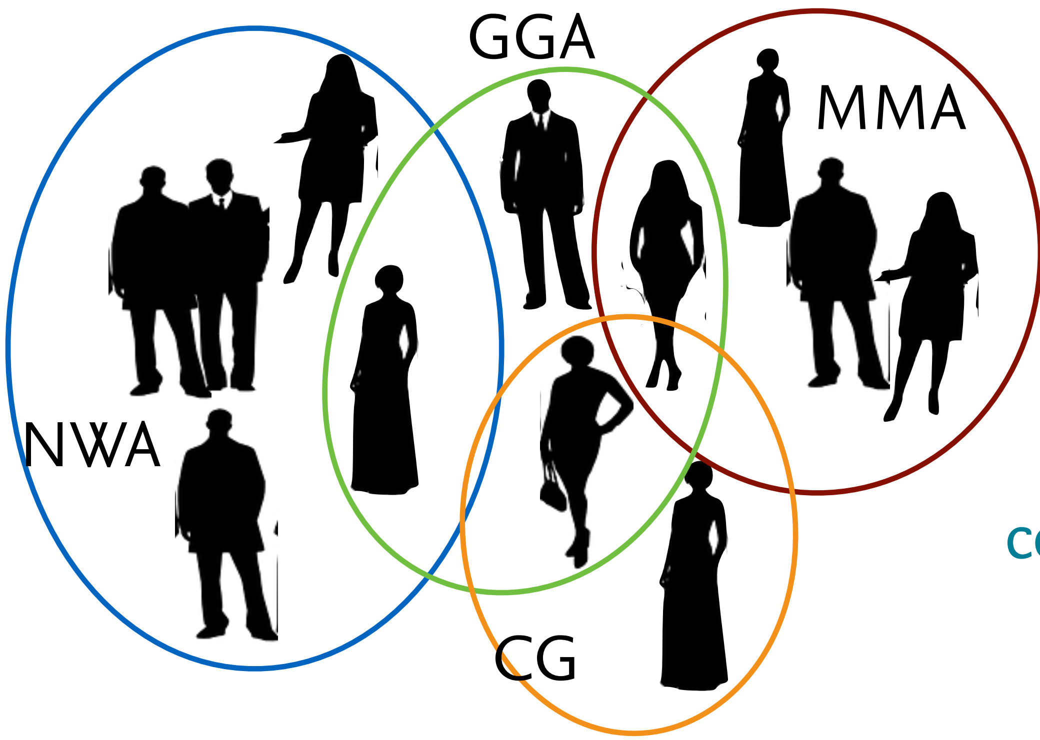


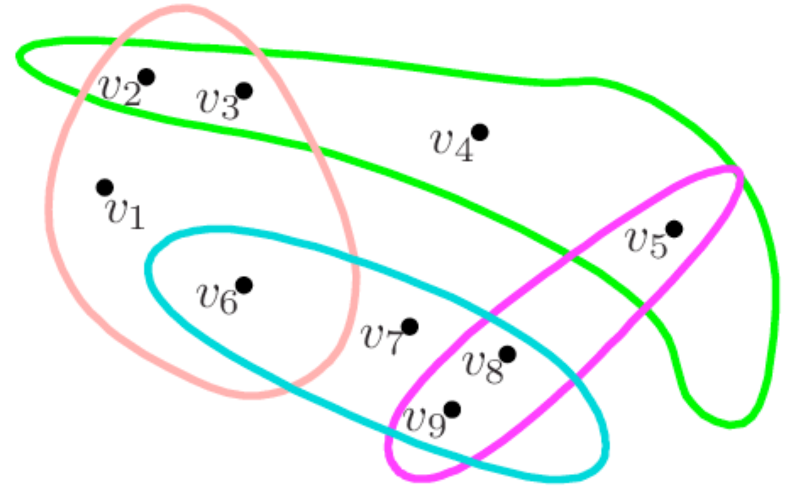




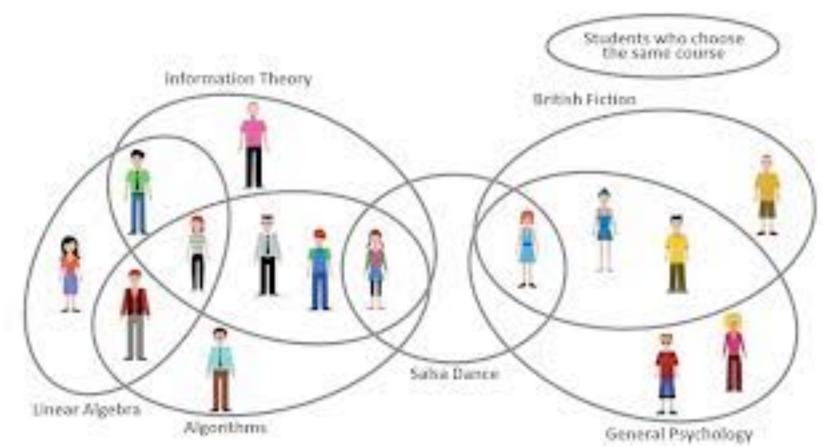


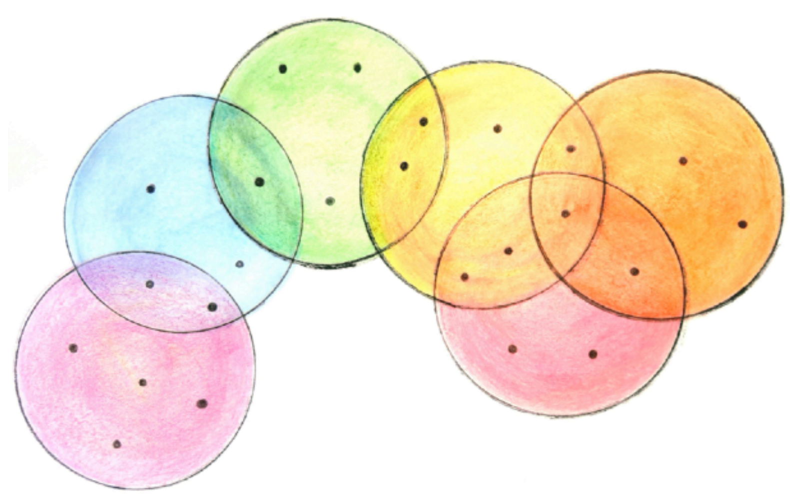
How to draw a hypergraph?



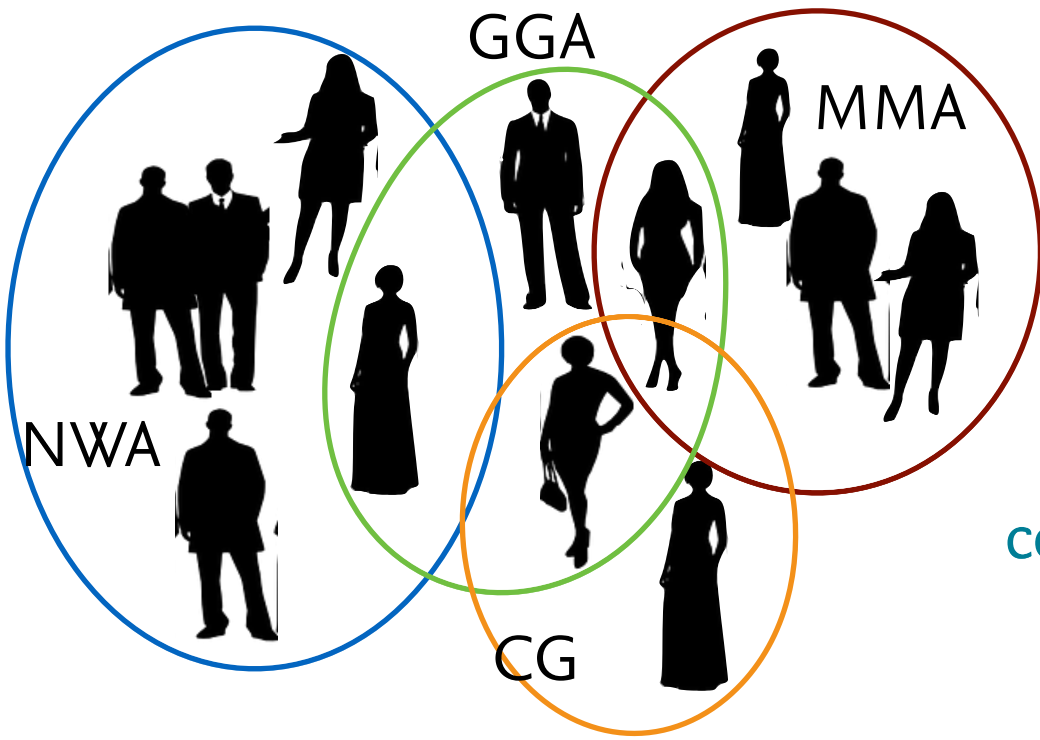


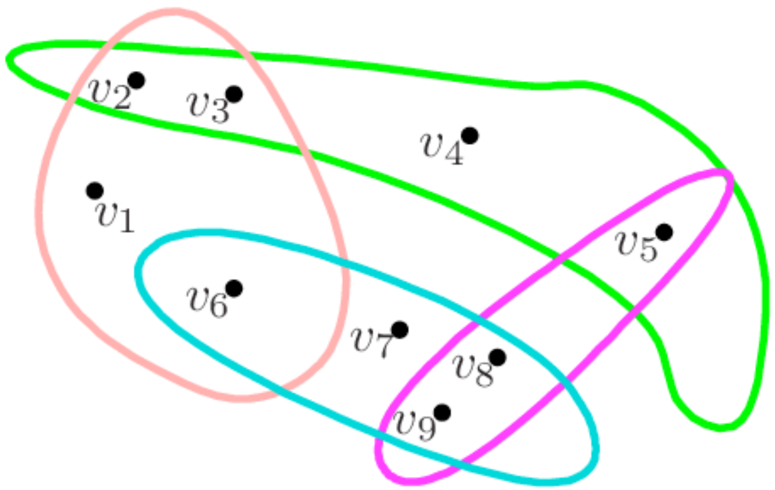
convex



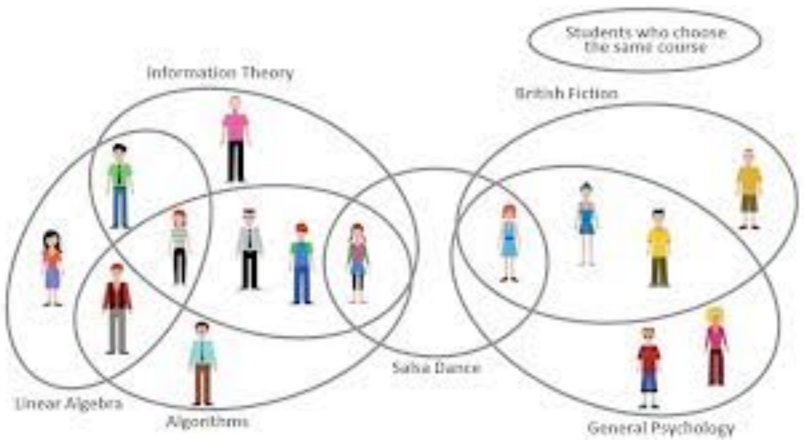


How to draw a hypergraph?

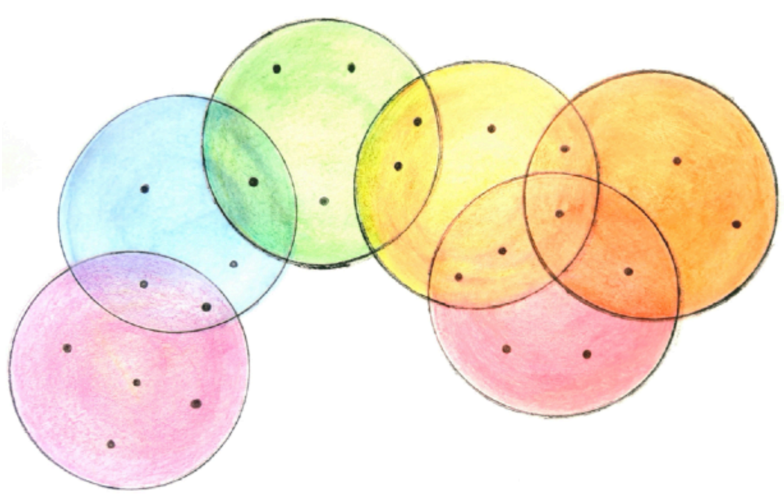




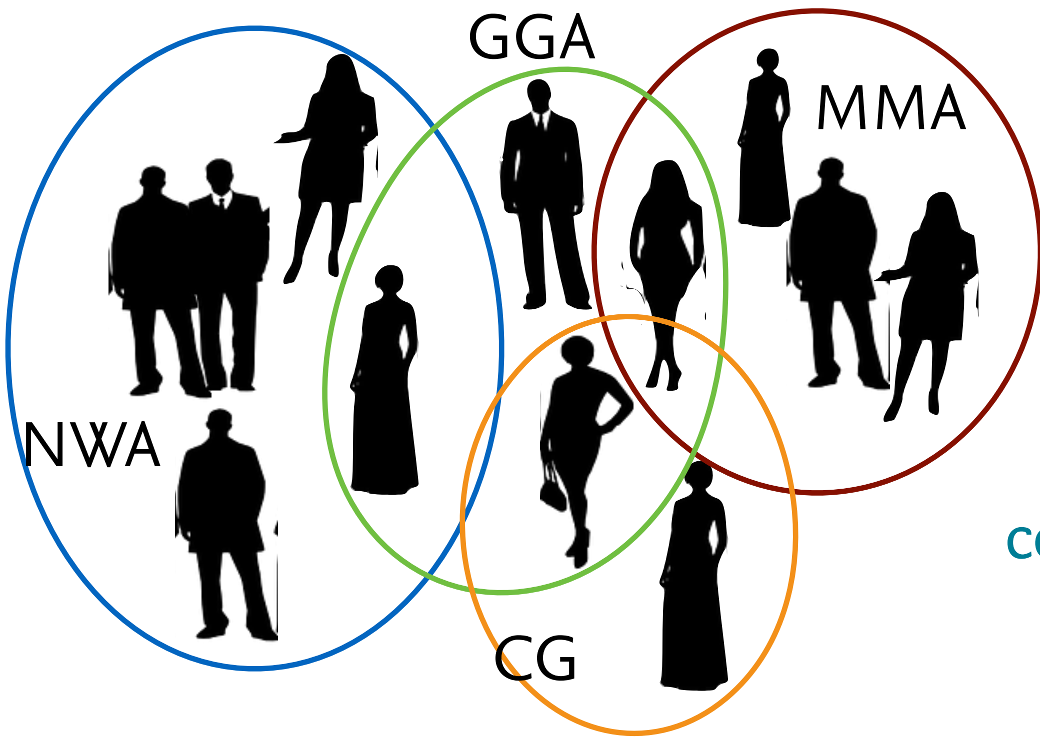
convex

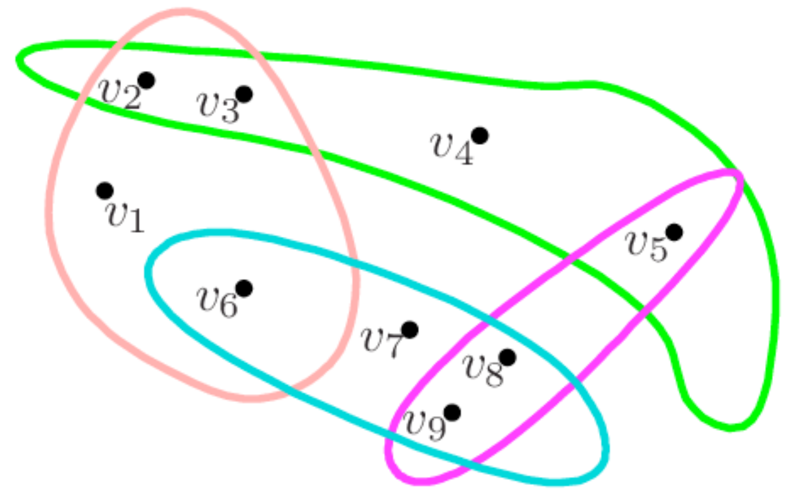


similar

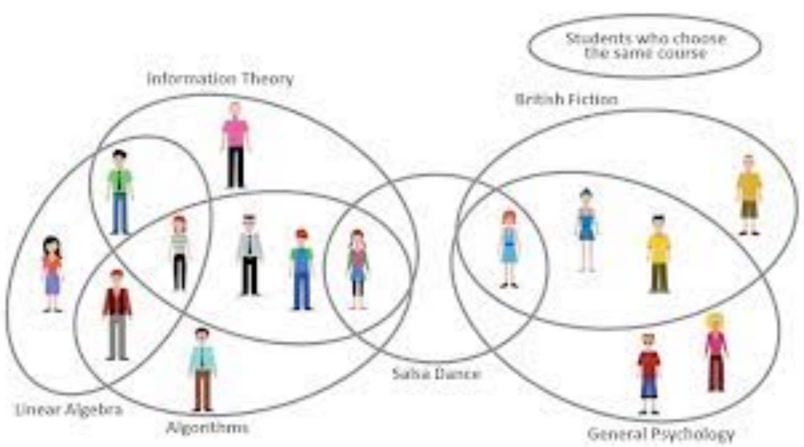


How to draw a hypergraph?

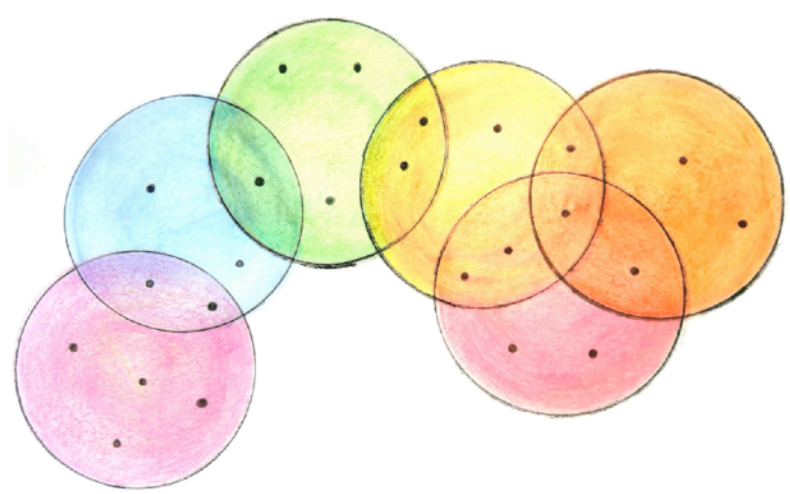




convex

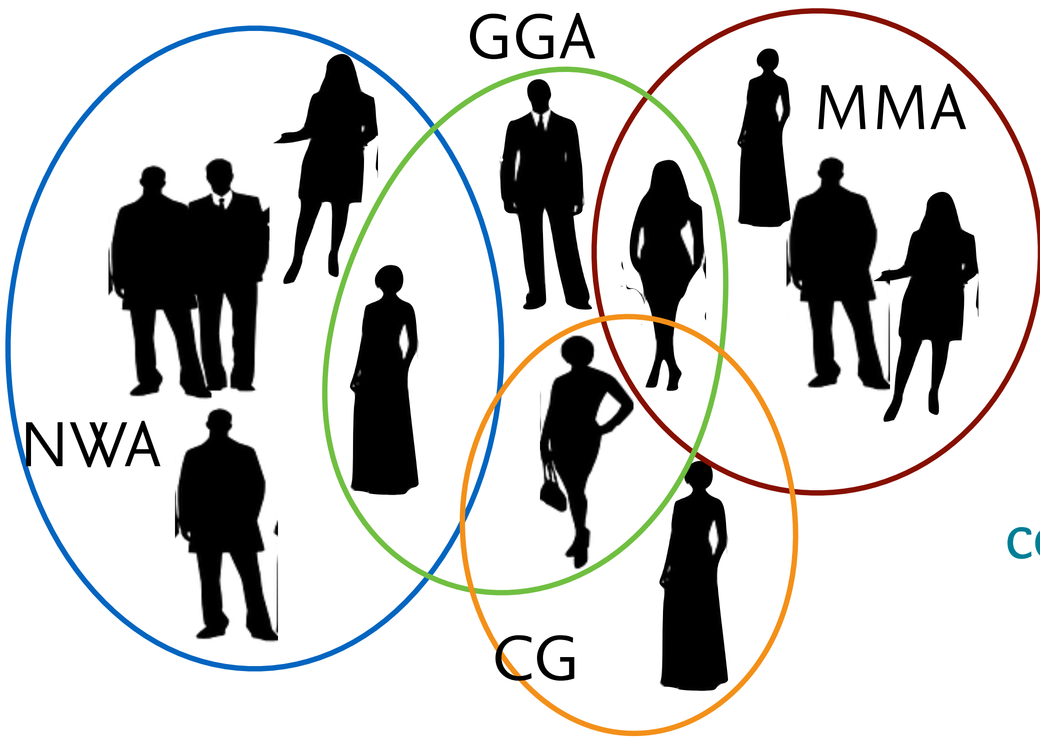


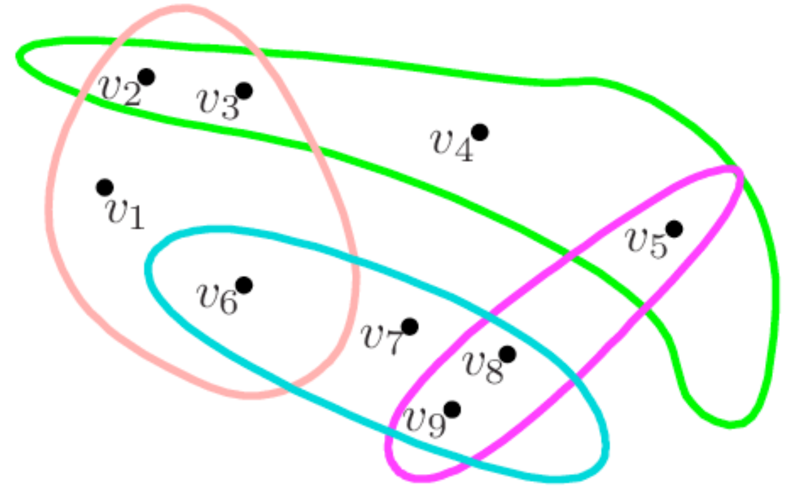
similar



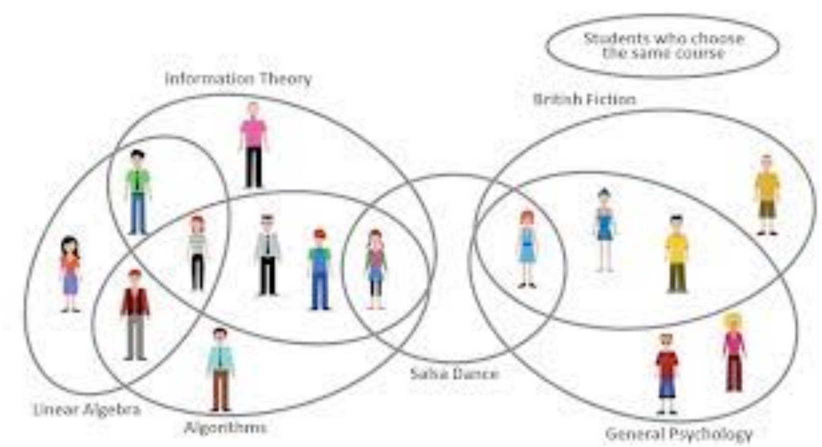
homothets

How to draw a hypergraph?

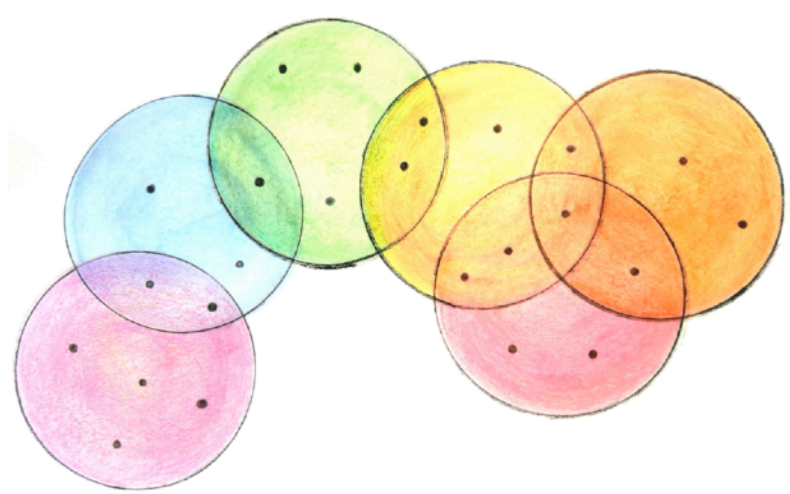




convex



similar



homothets

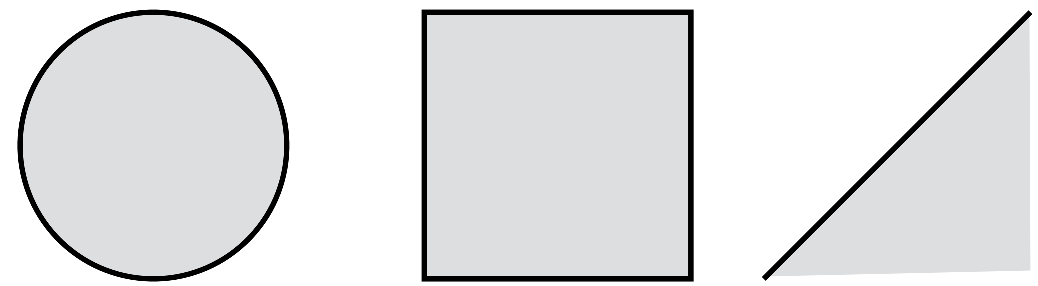
translates

The Problem: Recognition

$\mathcal{F}$ — family of sets in $\mathbb{R}^d$

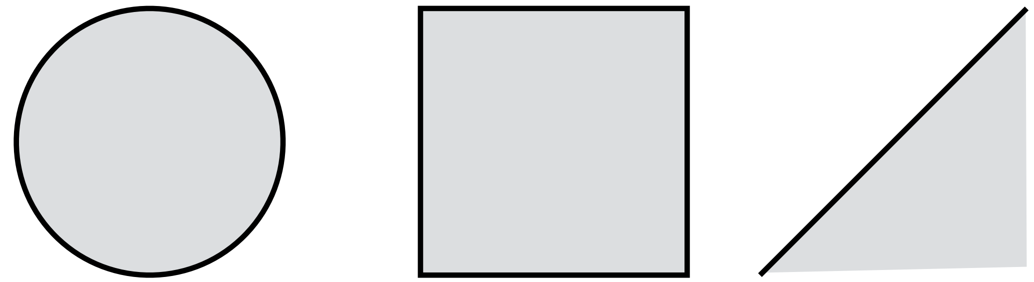
The Problem: Recognition

$\mathcal{F}$ — family of sets in $\mathbb{R}^d$



The Problem: Recognition

$\mathscr{F}$ — family of sets in $\mathbb{R}^d$

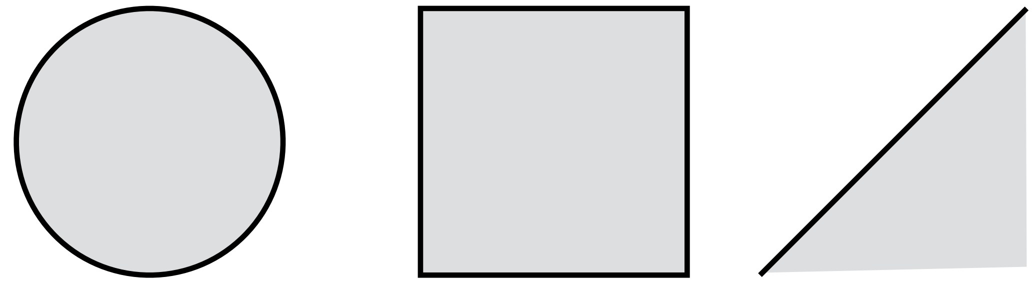


RECOGNITION($\mathscr{F}$)

Does a given hypergraph $H = (V, E)$ have an $\mathscr{F}$-representation?

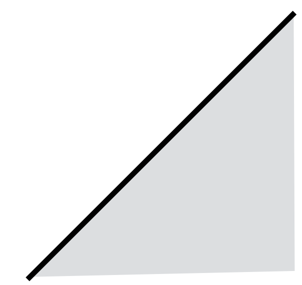
The Problem: Recognition

$\mathscr{F}$ — family of sets in $\mathbb{R}^d$



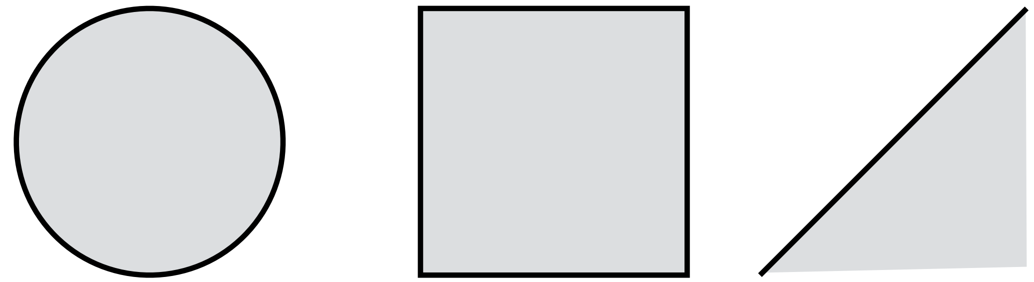
RECOGNITION($\mathscr{F}$)
Does a given hypergraph $H = (V, E)$ have an $\mathscr{F}$-representation?

- NP-hardness ($\exists\mathbb{R}$-completeness) for half spaces [Tanenbaum, Goodrich, Scheinerman, 1995]



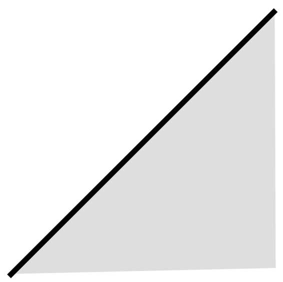
The Problem: Recognition

$\mathscr{F}$ — family of sets in $\mathbb{R}^d$



RECOGNITION($\mathscr{F}$)
Does a given hypergraph $H = (V, E)$ have an $\mathscr{F}$-representation?

- NP-hardness ($\exists\mathbb{R}$-completeness) for half spaces [Tanenbaum, Goodrich, Scheinerman, 1995]



Existential Theory of the Reals

Existential Theory of the Reals

NP ➤ SAT:

$\exists X = (X_1, X_2, \ldots, X_n) \in \{0,1\}^n \colon \Phi(X)$

operations $=, \wedge, \vee, \neg$

Existential Theory of the Reals

NP ➤ SAT:

$\exists X = (X_1, X_2, \ldots, X_n) \in \{0,1\}^n : \Phi(X)$

operations $=, \wedge, \vee, \neg$

$\exists\mathbb{R}$ ➤ ETR:

$\exists X = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^n : \Phi(X)$

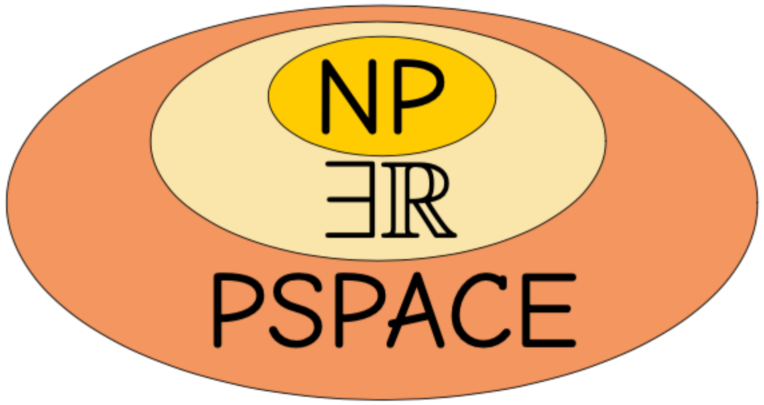
operations: $=, <, \leq, +, \cdot, \wedge, \vee, \neg$

Existential Theory of the Reals

NP ➤ SAT:

$\exists X = (X_1, X_2, \ldots, X_n) \in \{0,1\}^n \colon \Phi(X)$

operations $=, \wedge, \vee, \neg$



∃ℝ ➤ ETR:

$\exists X = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^n \colon \Phi(X)$

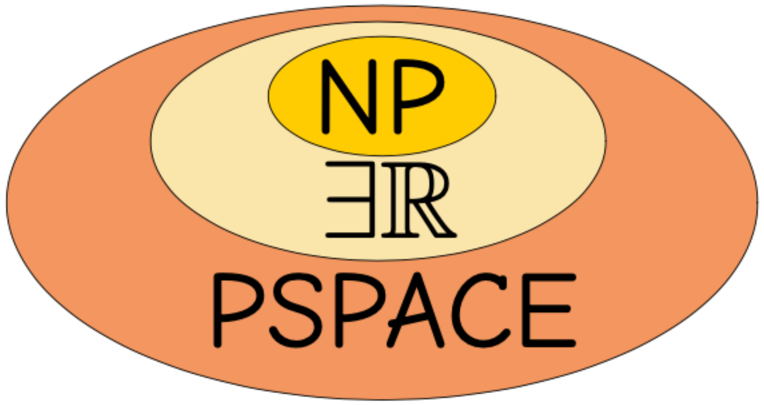
operations: $=, <, \leq, +, \cdot, \wedge, \vee, \neg$

Existential Theory of the Reals

NP ➤ SAT:

$$\exists X = (X_1, X_2, \ldots, X_n) \in \{0,1\}^n : \Phi(X)$$

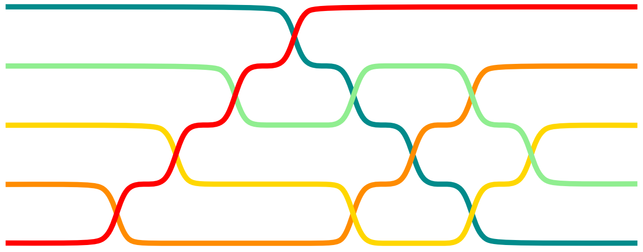
operations $=, \wedge, \vee, \neg$

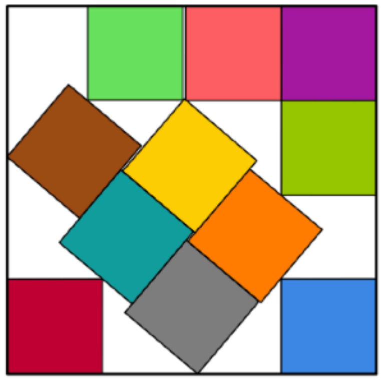


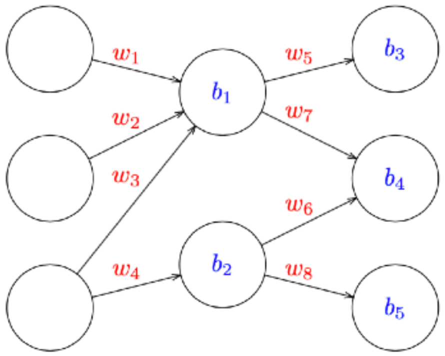
∃ℝ ➤ ETR:

$$\exists X = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^n : \Phi(X)$$

operations: $=, <, \leq, +, \cdot, \wedge, \vee, \neg$







Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

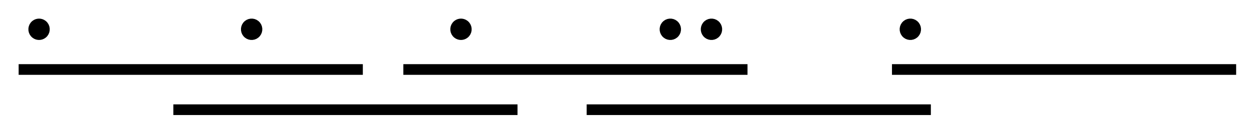
- $d = 1$

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$

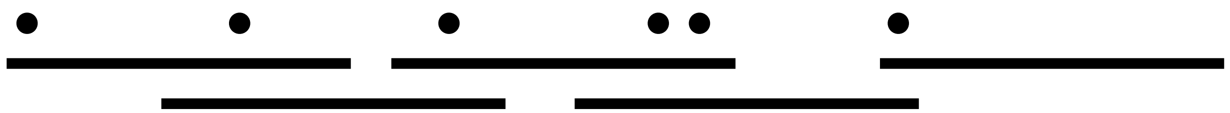


Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that Recognition($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ … in P

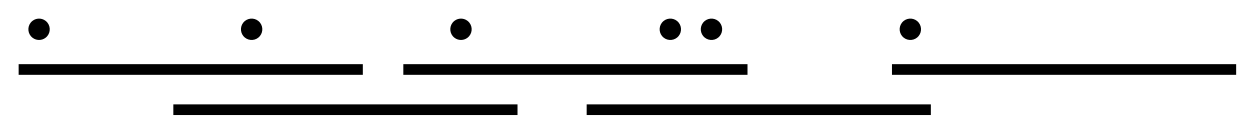


Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P
- polygon?

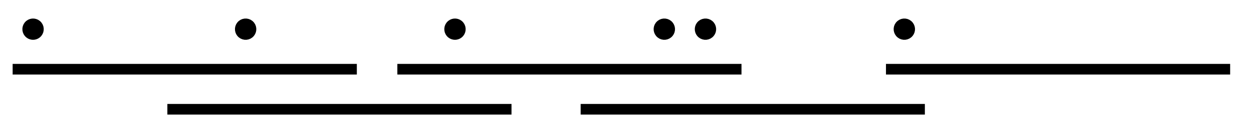


Translates

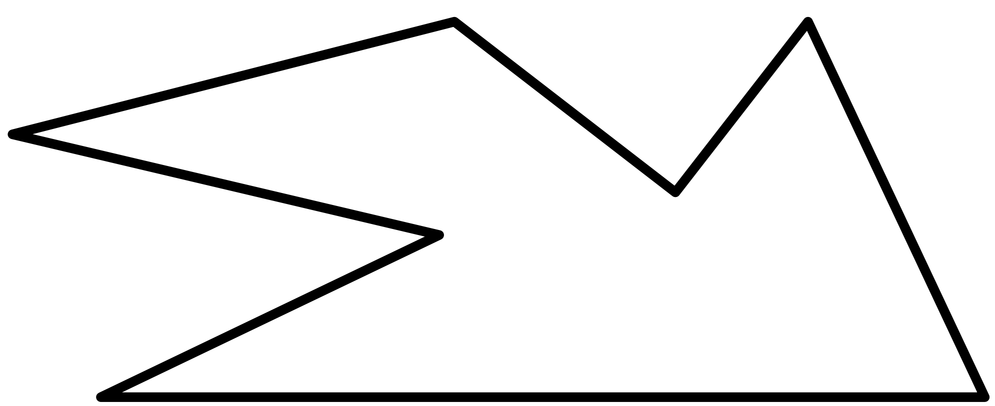
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P



- polygon?

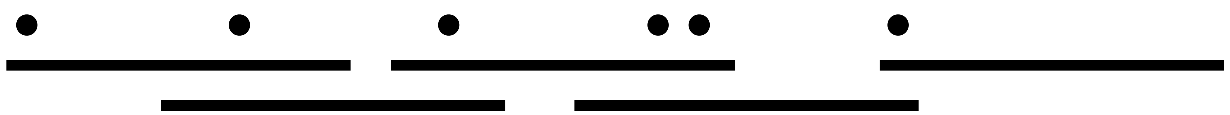


Translates

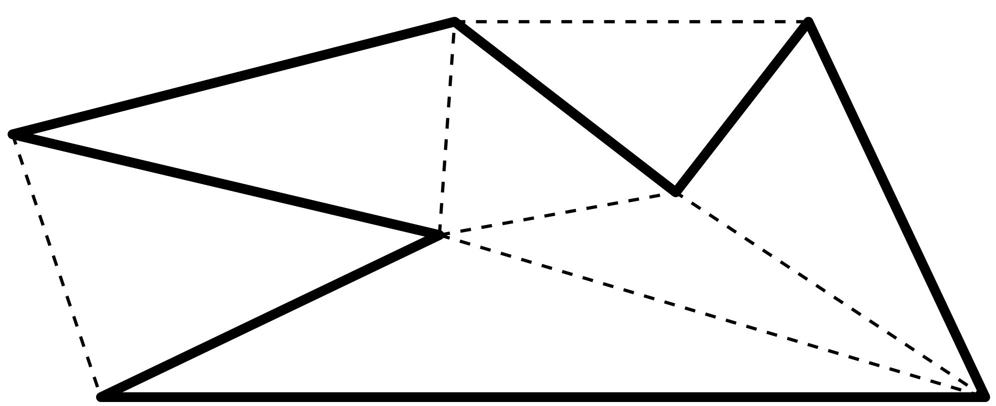
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P



- polygon?

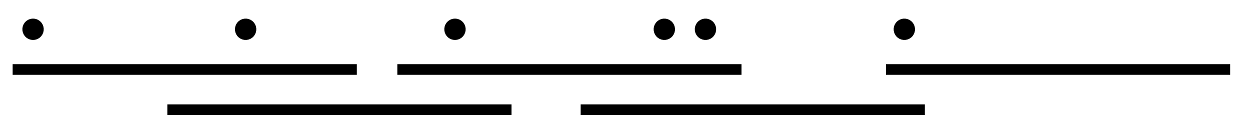


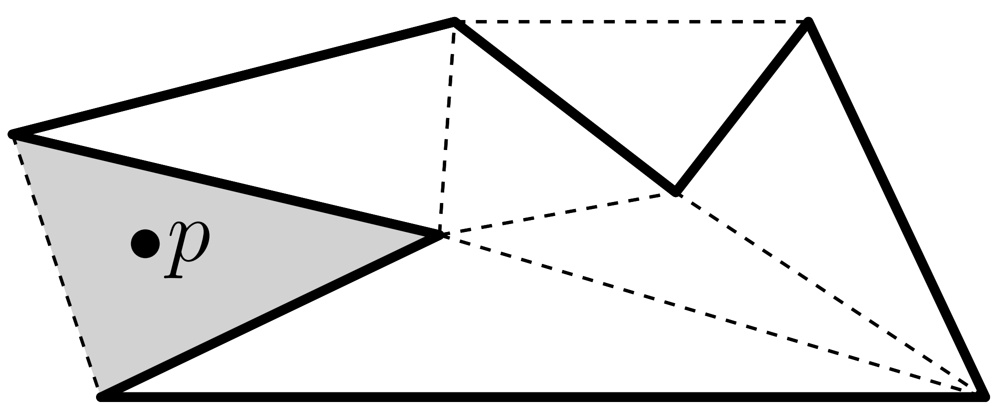
Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ … in P
- polygon?



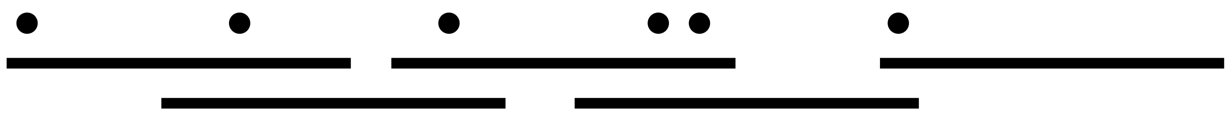


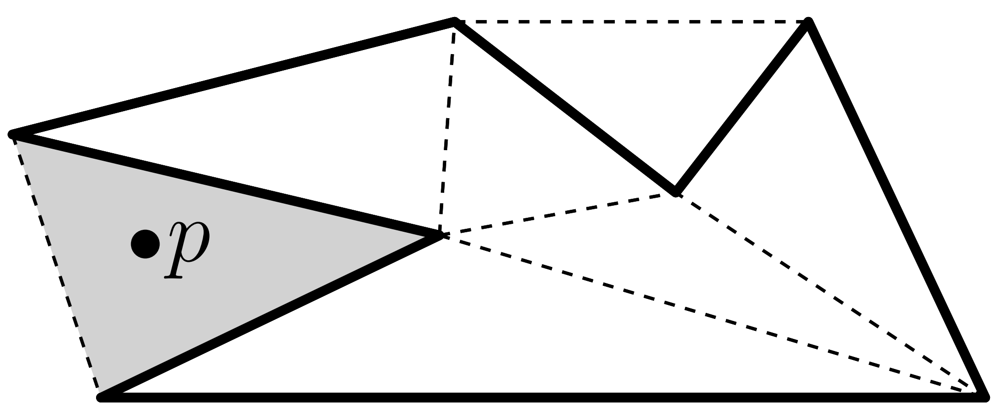
Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P
- polygon?





CERTIFICATE

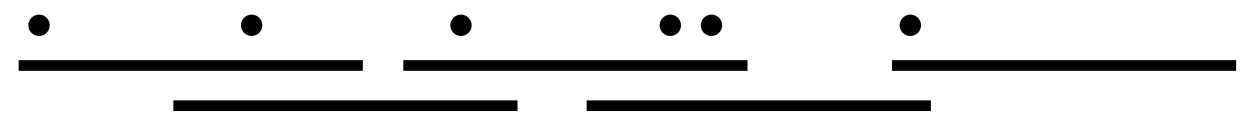
for each point and translate,
specify triangle or sep edge

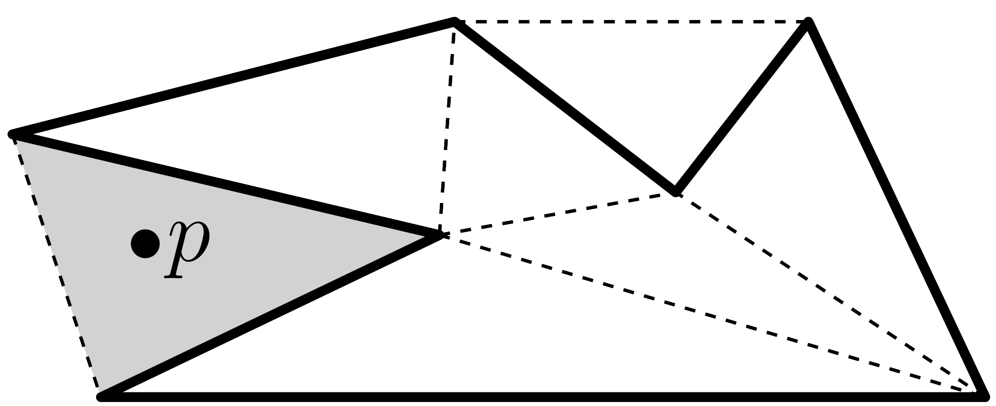
Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ … in P
- polygon?





CERTIFICATE

for each point and translate, specify triangle or sep edge

point coordinates

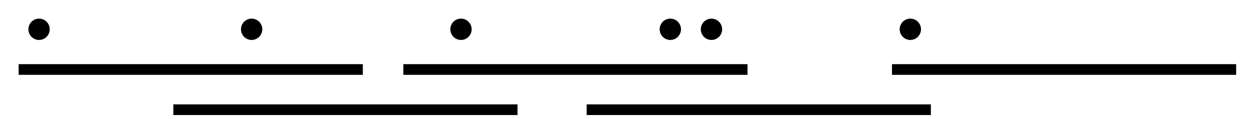
LP translation vectors

Translates

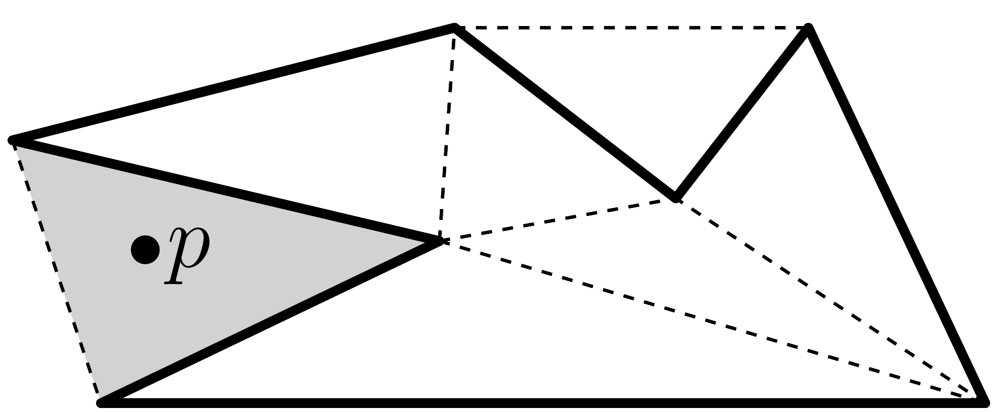
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ … in P



- polygon?



CERTIFICATE

for each point and translate,
specify triangle or sep edge

LP

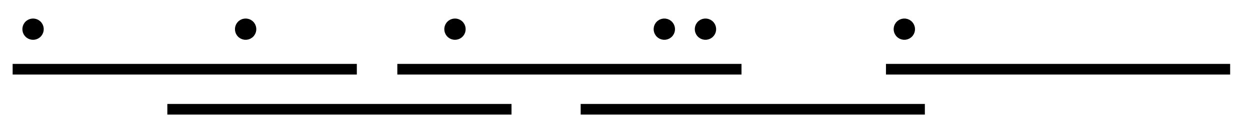
point coordinates → variables
translation vectors → variables

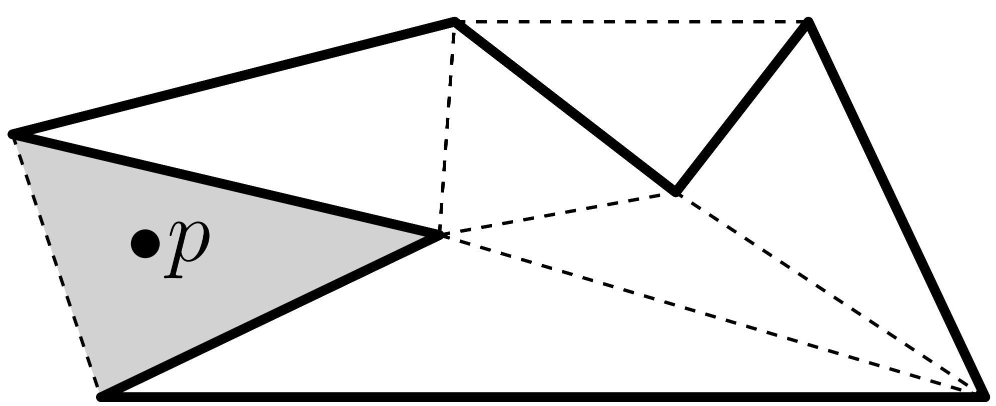
Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P
- polygon?





CERTIFICATE

for each point and translate, specify triangle or sep edge

LP

point coordinates → variables

translation vectors → variables

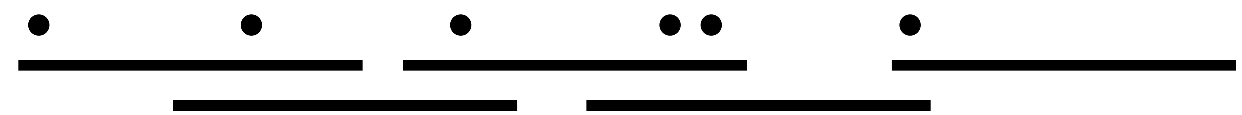
incidences → constraints

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

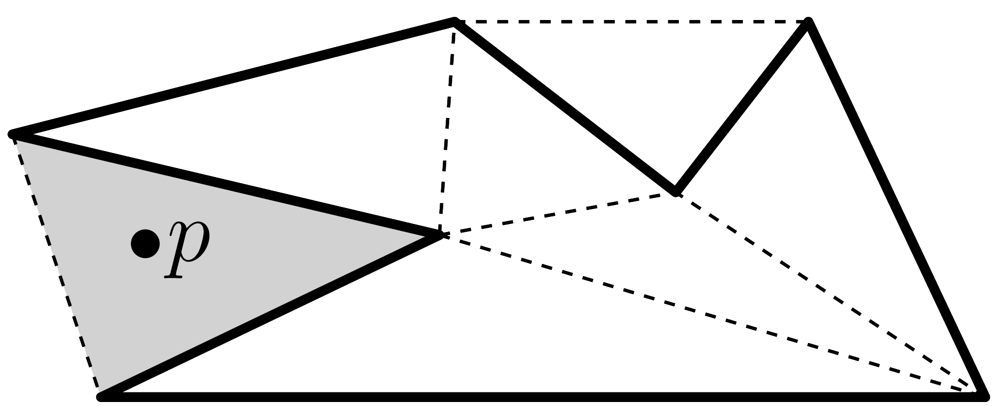
- $d = 1$ ... in P



- polygon?

Theorem

If C is a simple polygon in $\mathbb{R}^2$ then RECOGNITION($\mathbb{T}_C$) is in NP.



CERTIFICATE

for each point and translate, specify triangle or sep edge

LP

point coordinates → variables

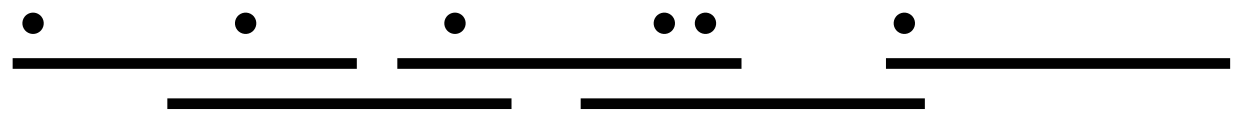
translation vectors → variables

incidences → constraints

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?



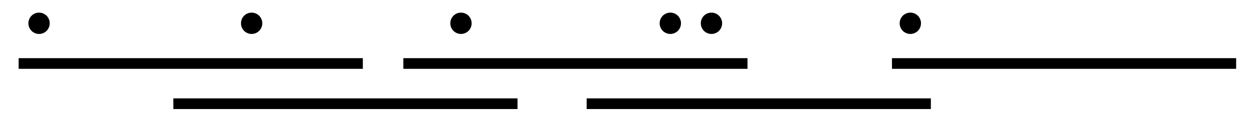
- $d = 1$ ... in P
- polygon? ... in NP

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P



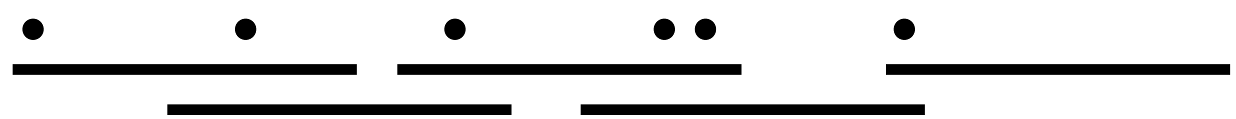
- polygon? ... in NP $\implies$ some curvature is 'necessary'

Translates

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

What properties of C imply that RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete?

- $d = 1$ ... in P
- polygon? ... in NP

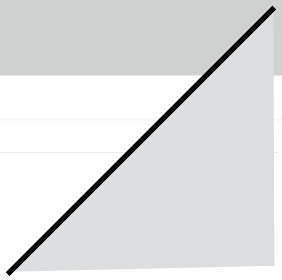


$\implies$ some curvature is 'necessary'

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Halfspaces - Warm up



Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Halfspaces - Warm up

Theorem [TGS, 1995]

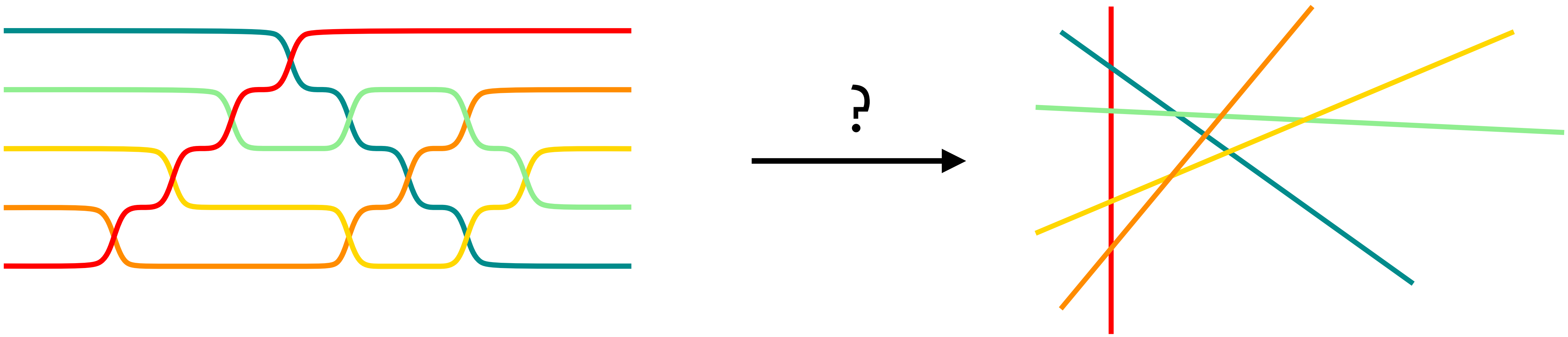
For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

Halfspaces - Warm up

Theorem [TGS, 1995]
For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.



pseudoline arrangement

combinatorial equivalent
line arrangement

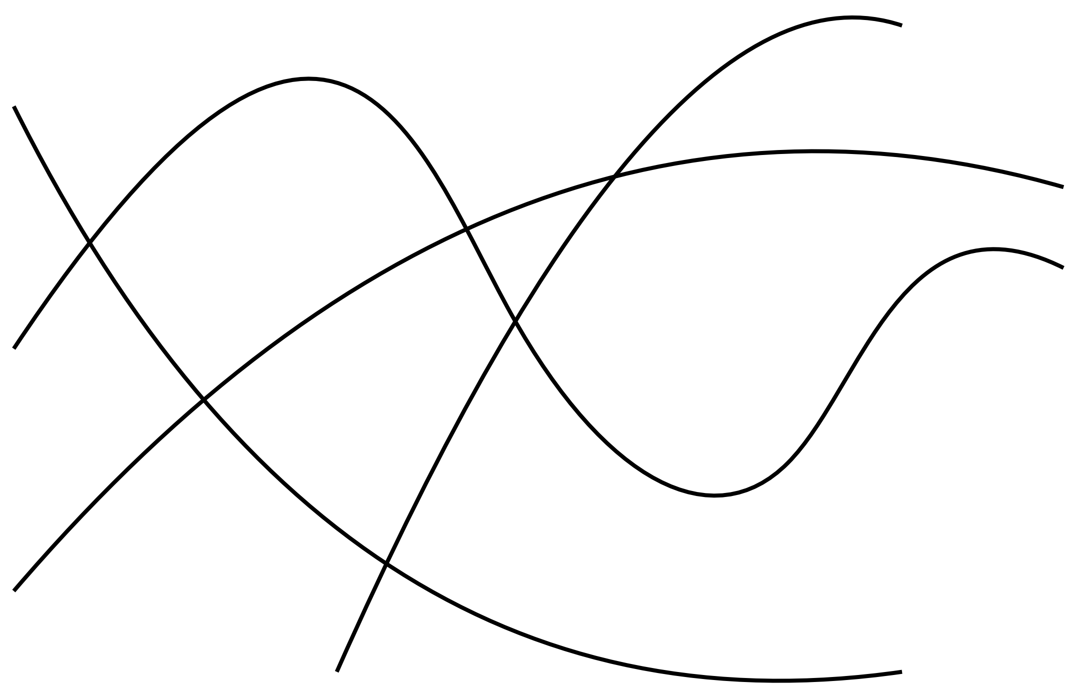
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

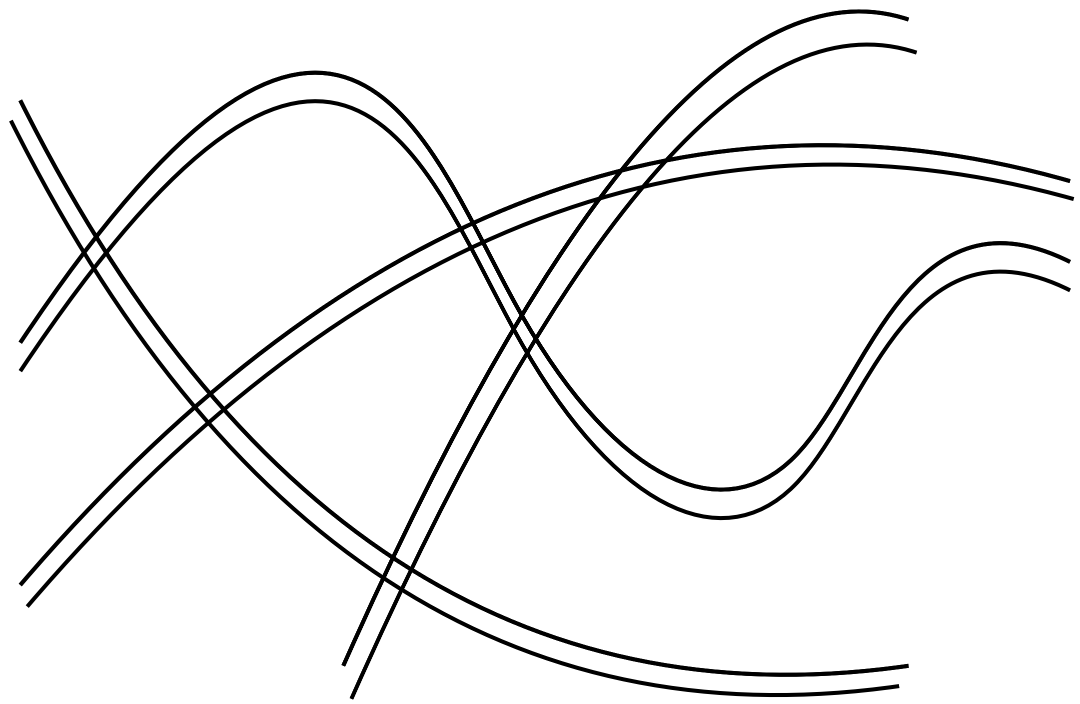
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

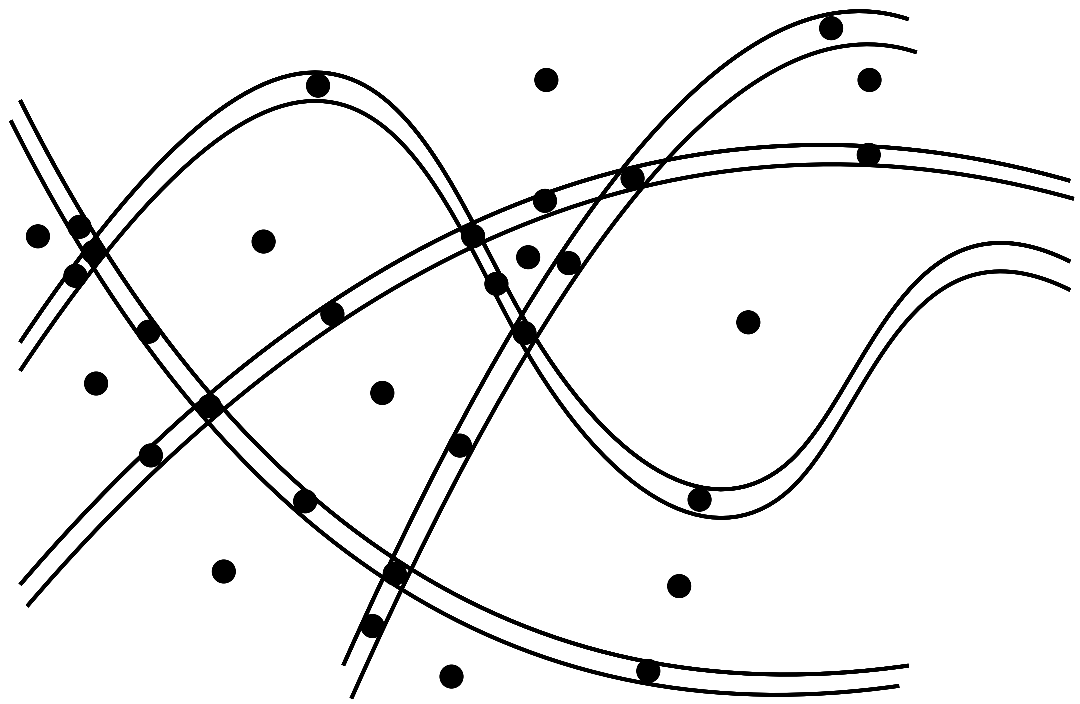
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

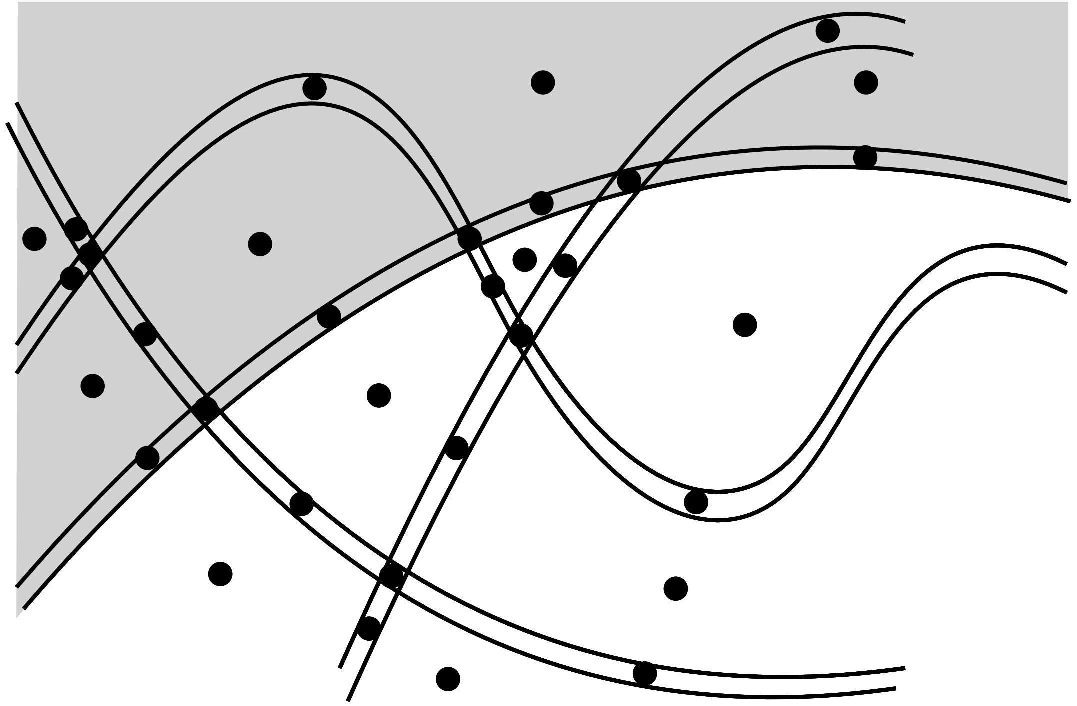
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

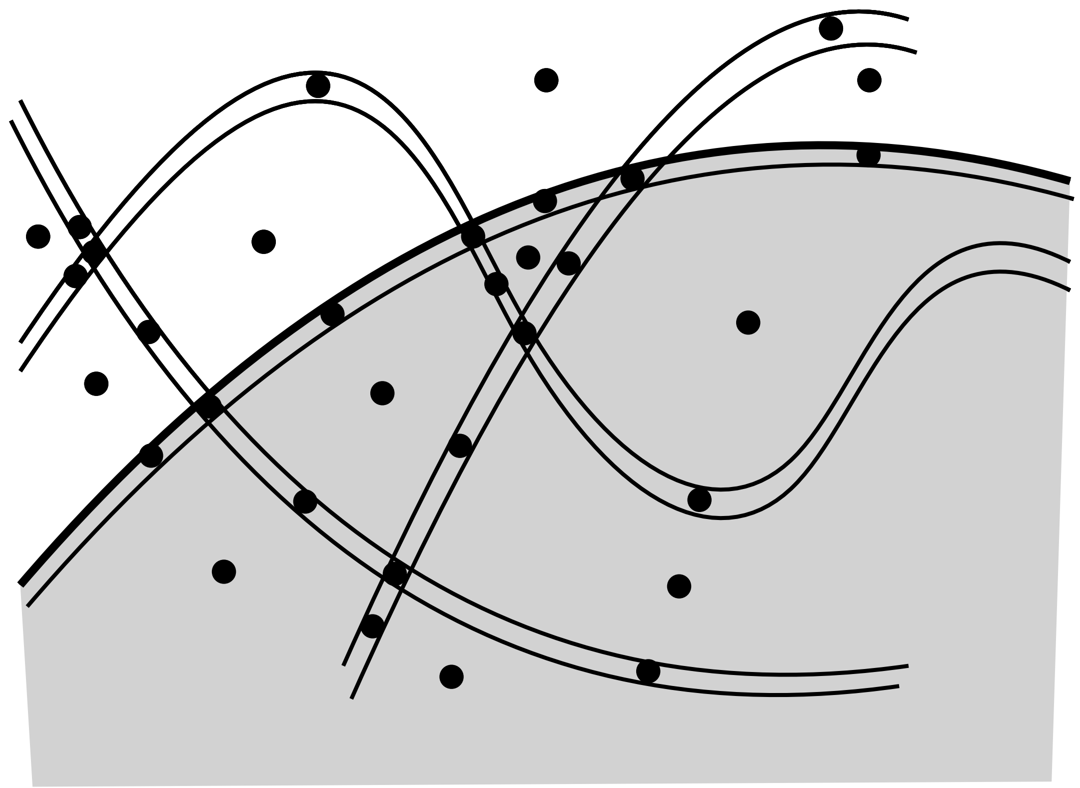
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

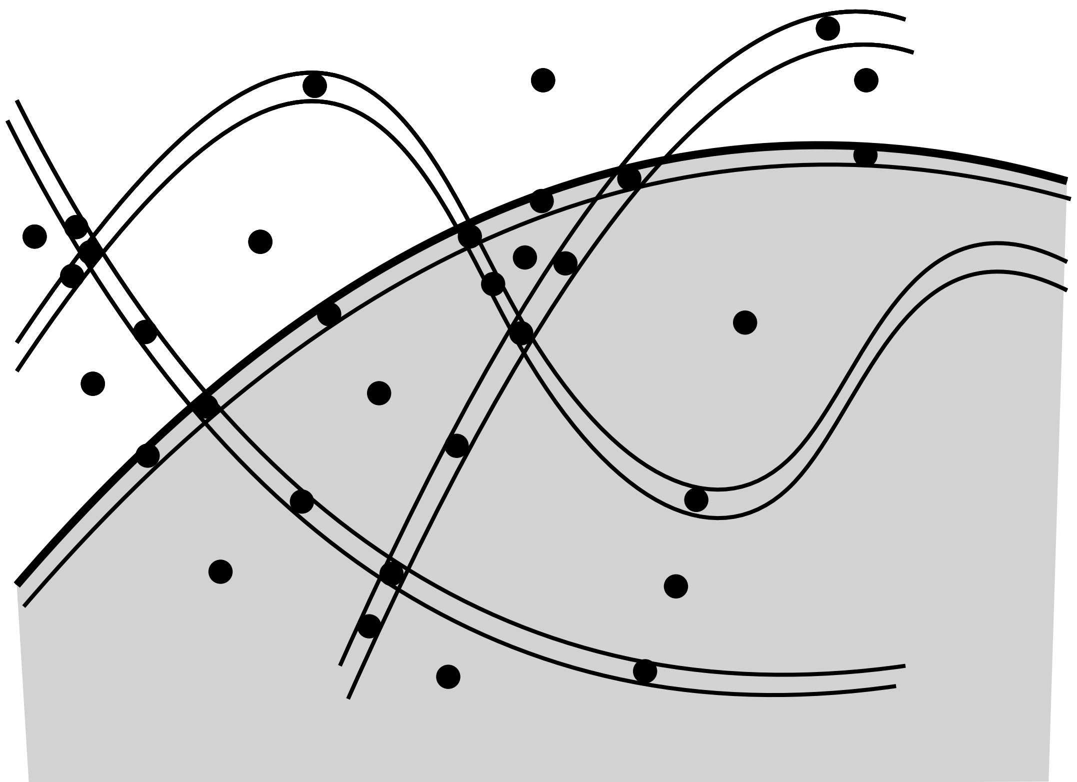
Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation

$\mathscr{A}$



$H = (V, E)$

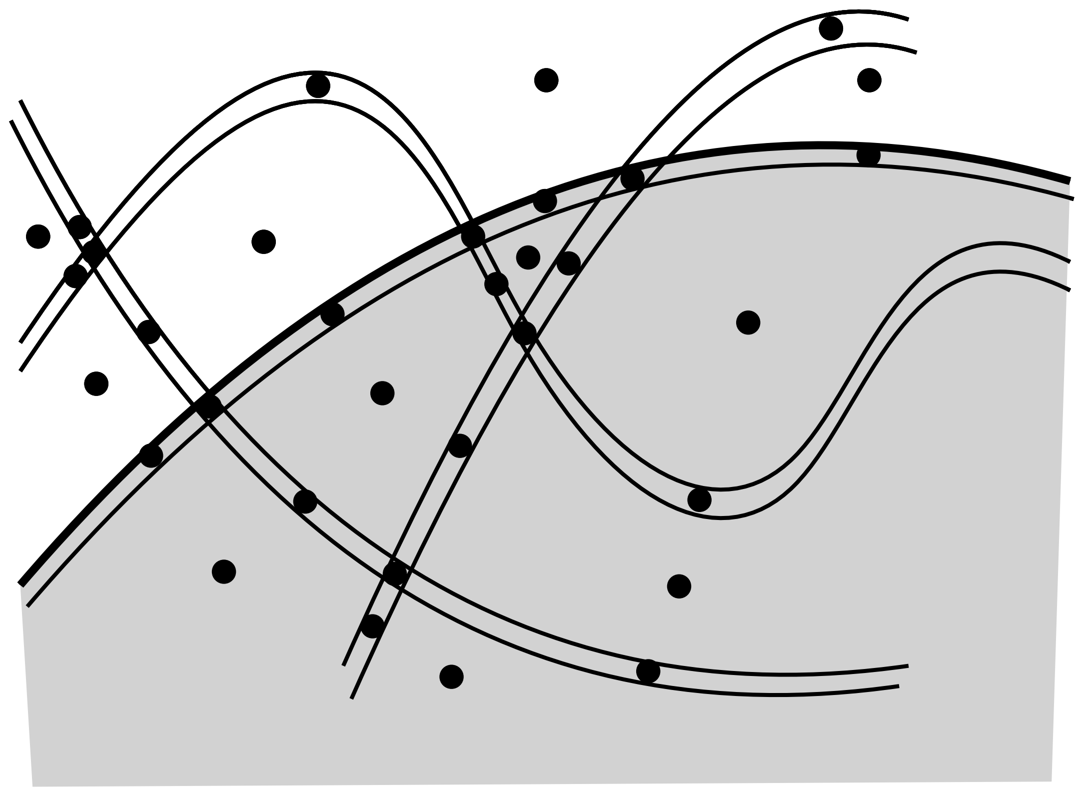
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

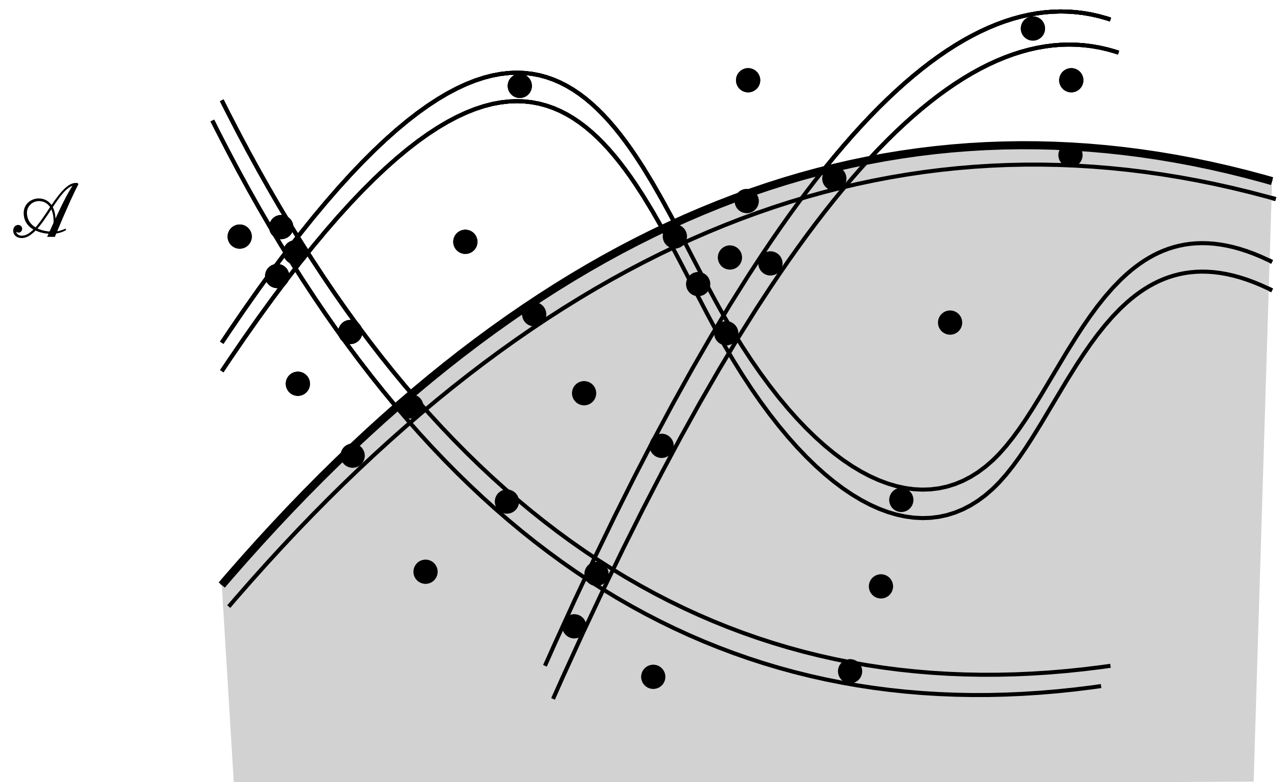
- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation

Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.



$H = (V, E)$

- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation
- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable

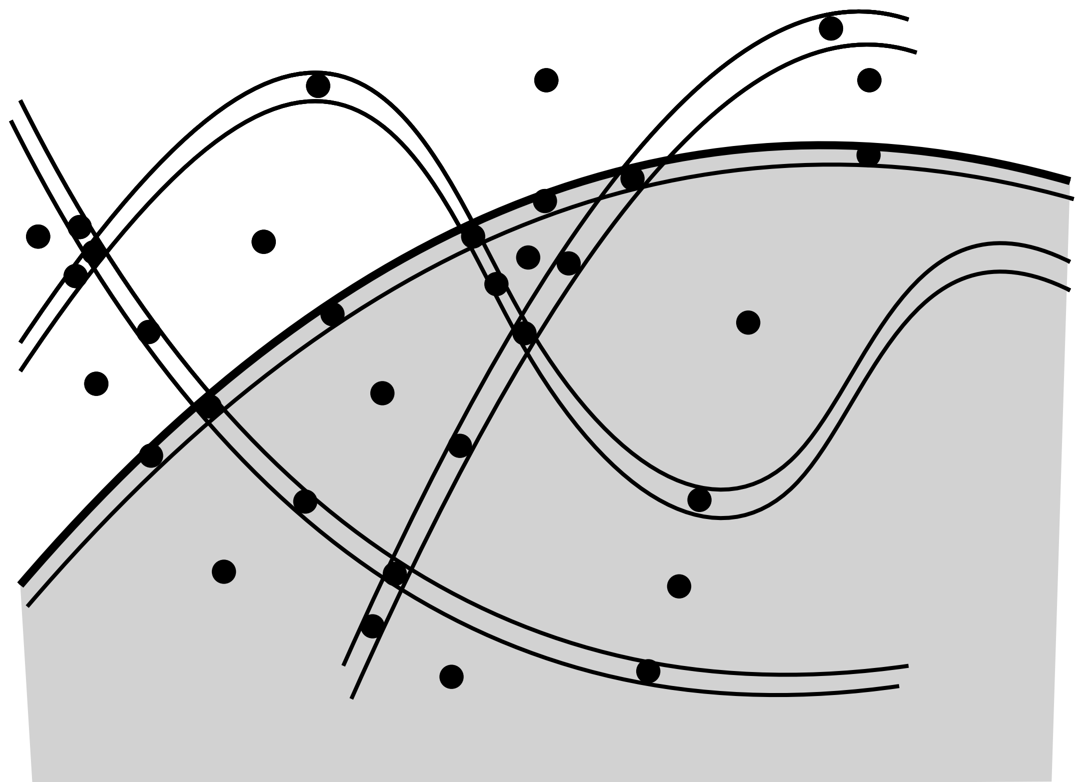
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

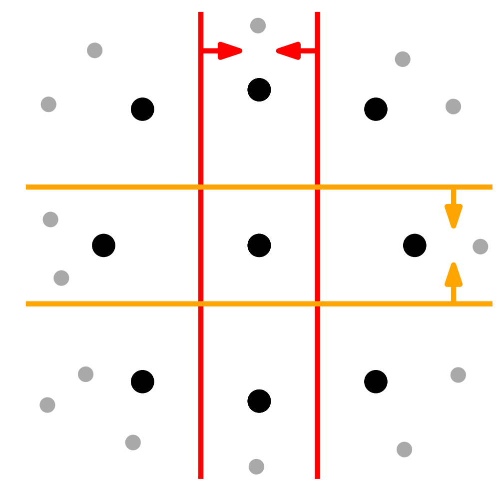
Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation
- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable



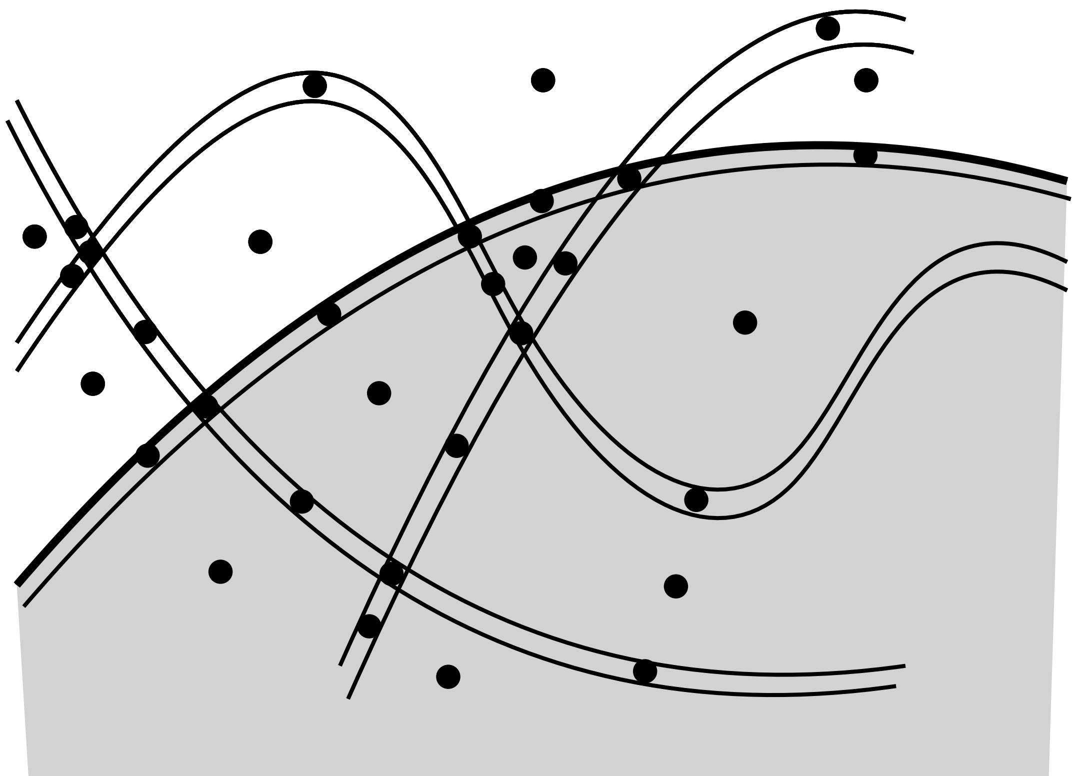
Halfspaces - Warm up

Theorem [TGS, 1995]

For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

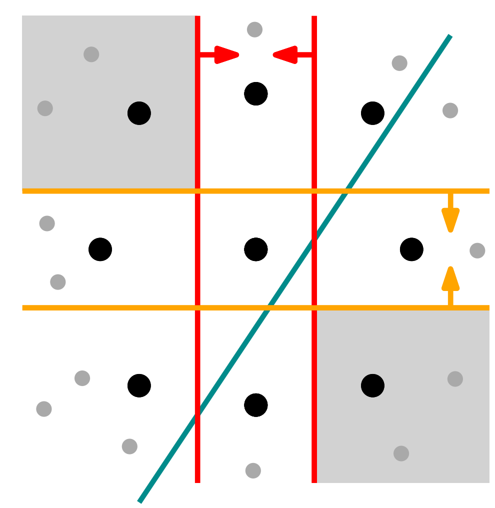
Proof*: Reduction from STRETCHABILITY.

$\mathscr{A}$



$H = (V, E)$

- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation
- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable

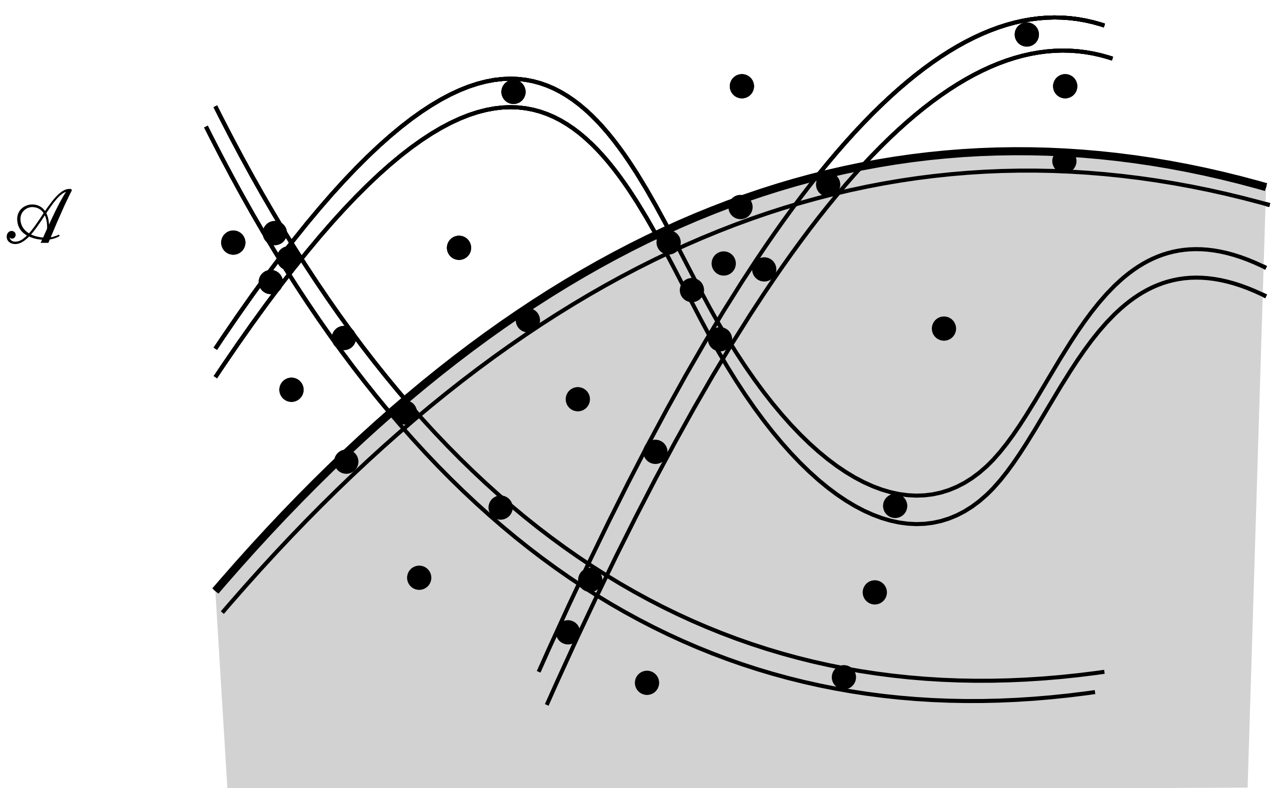


Halfspaces - Warm up

Theorem [TGS, 1995]

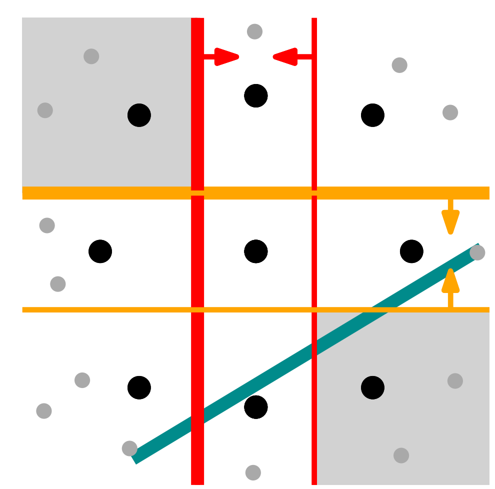
For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.



$H = (V, E)$

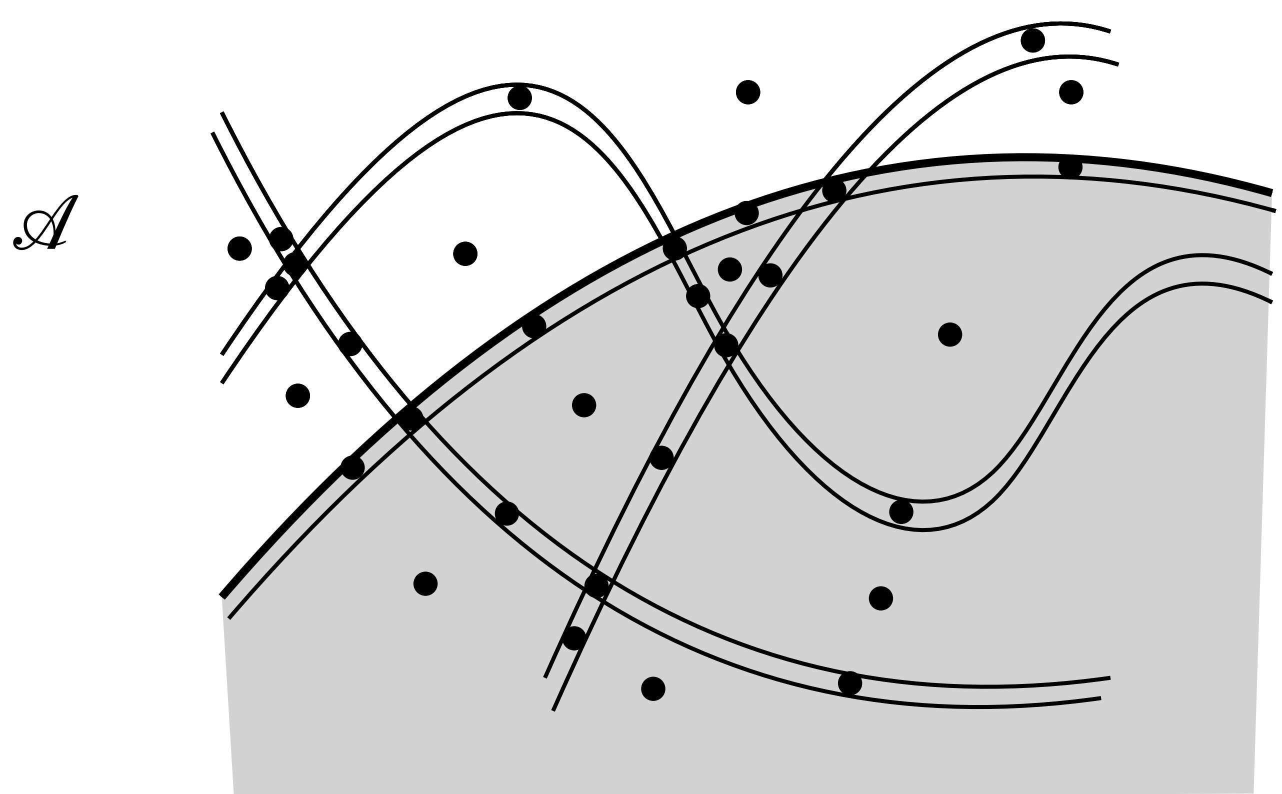
- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation
- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable



Theorem [TGS, 1995]

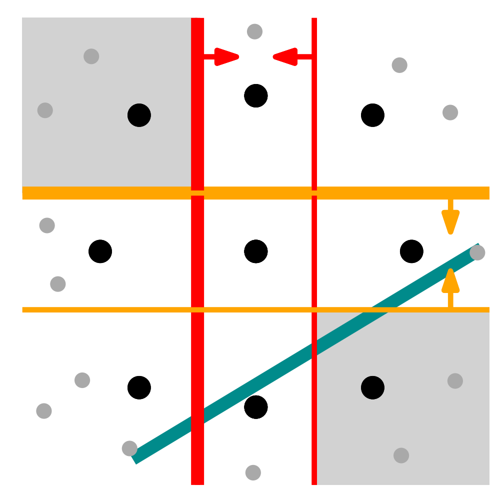
For the family $\mathscr{F}$ of halfspaces, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof*: Reduction from STRETCHABILITY.

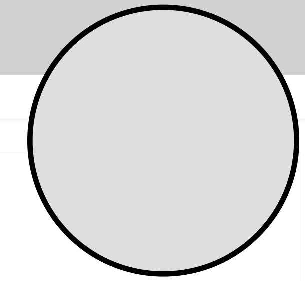


$$H = (V, E)$$

- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation
- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable



Unit disks



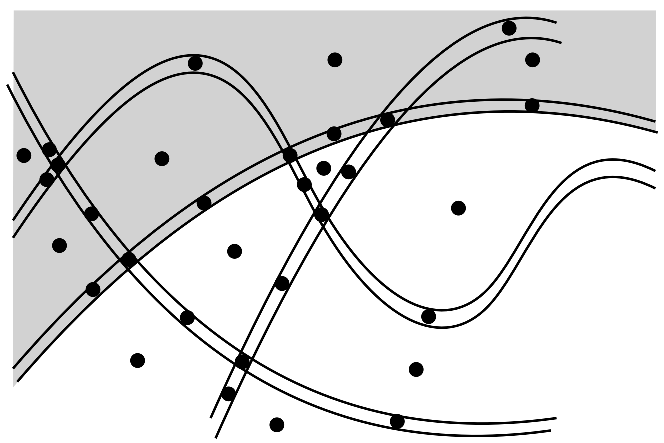
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof: Reduction form STRETCHABILITY.

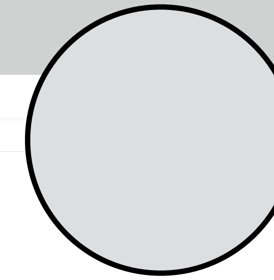
$\mathscr{A}$

$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation

Unit disks



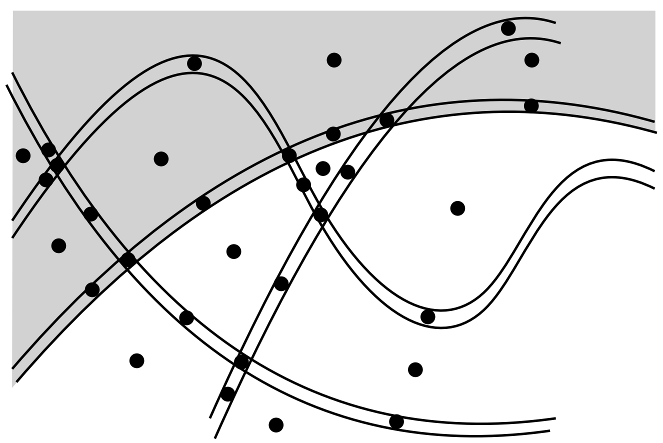
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

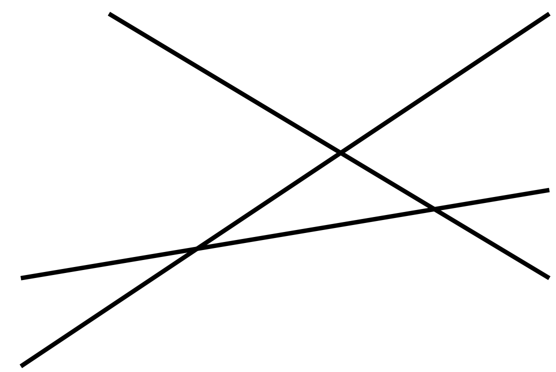
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

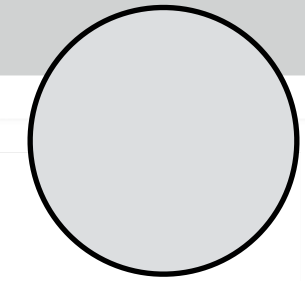
$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



Unit disks



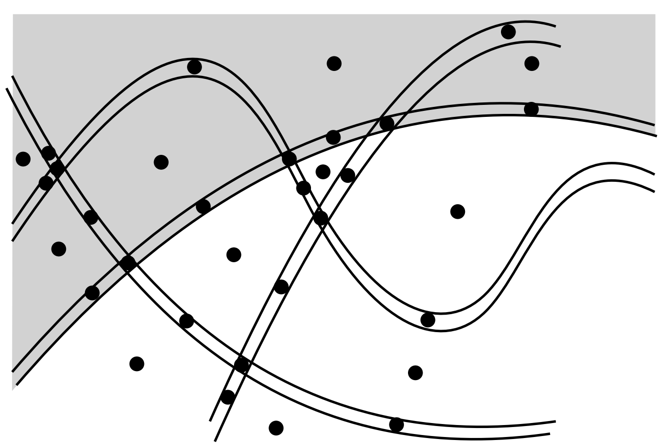
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

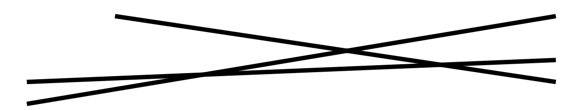
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

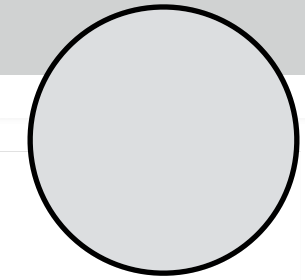
$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



Unit disks



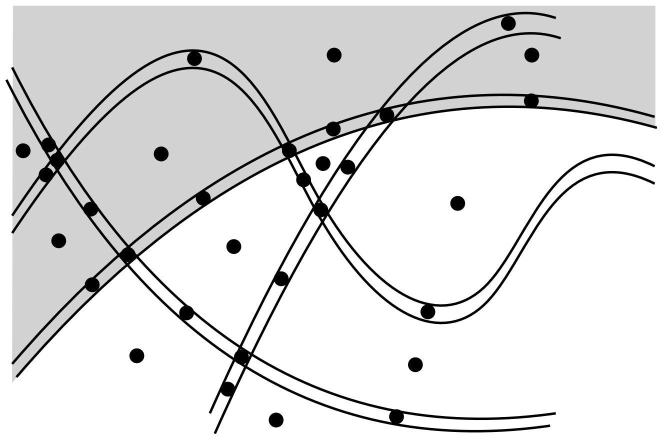
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

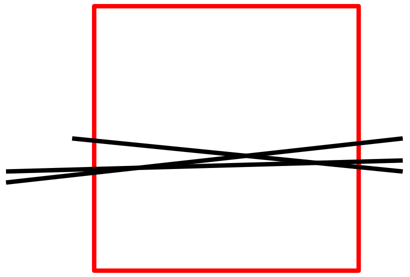
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

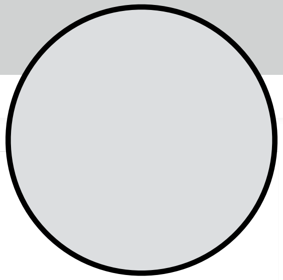
$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



Unit disks



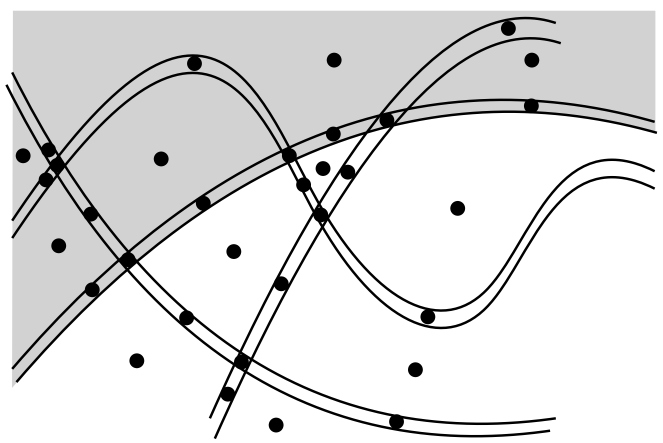
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

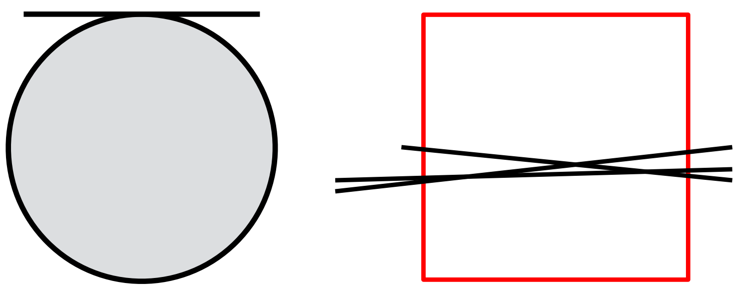
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

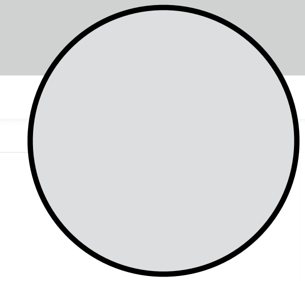
$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



Unit disks



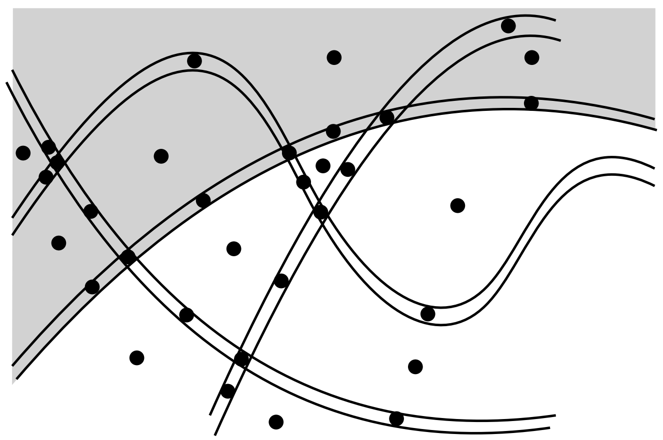
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

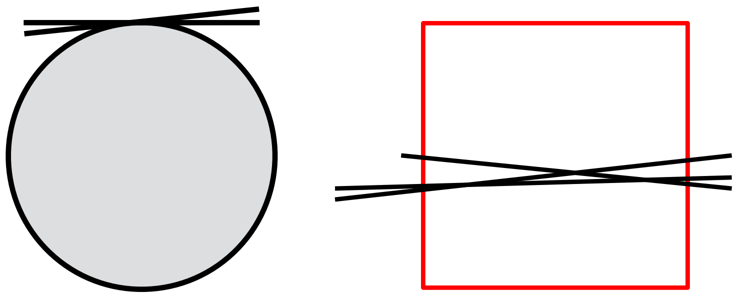
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

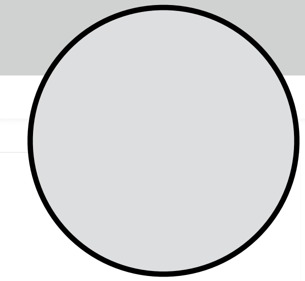
$H = (V, E)$



- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



Unit disks



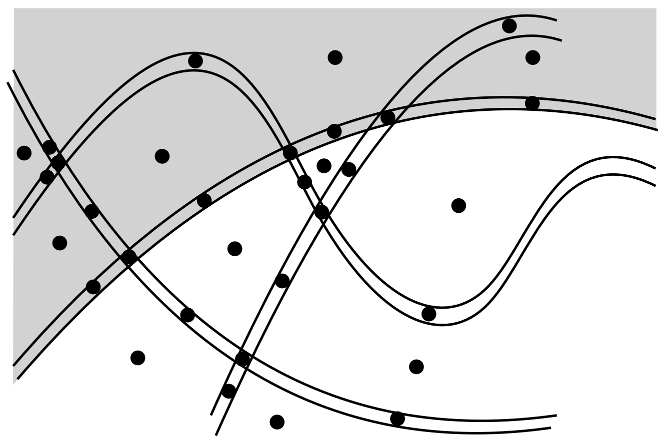
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

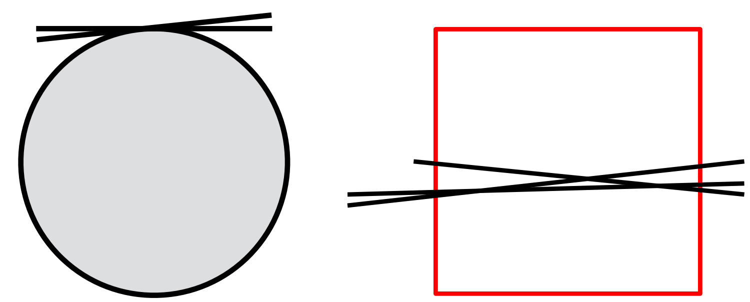
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

$H = (V, E)$

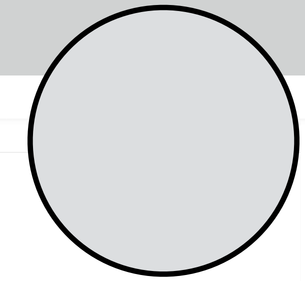


- $\mathscr{A}$ stretchable $\implies$ H has $\mathscr{F}$-representation



- $\mathscr{F}$-representation of $H \implies$ $\mathscr{A}$ stretchable

Unit disks



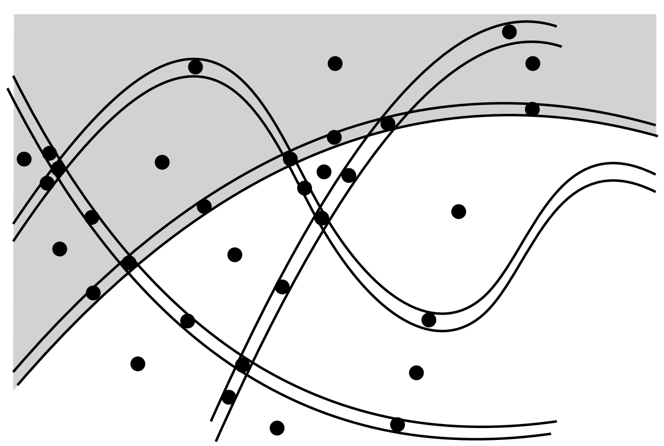
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

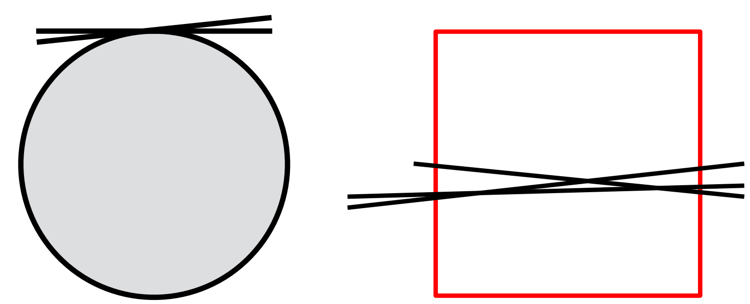
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

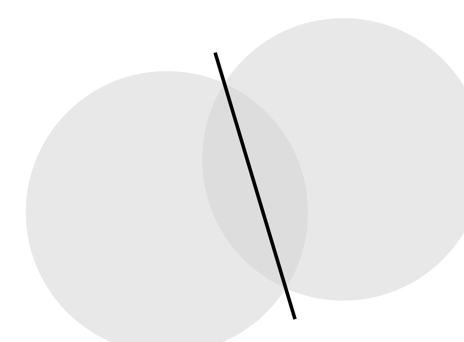
$H = (V, E)$

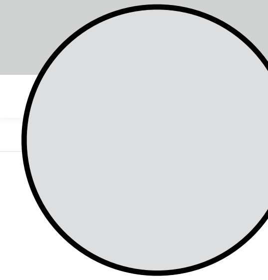


- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable





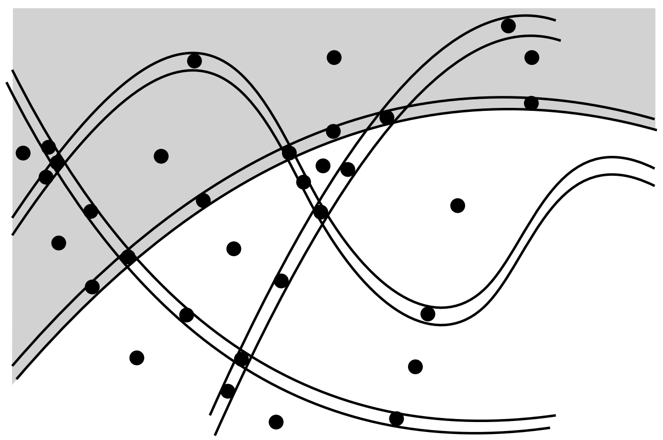
Theorem

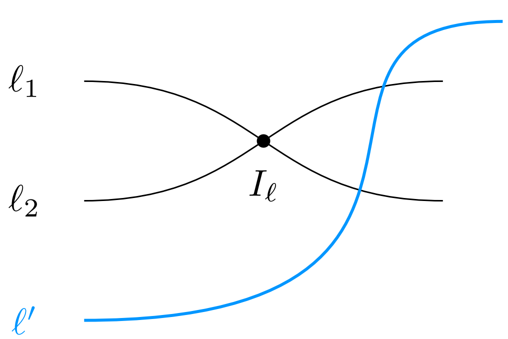
For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof: Reduction form STRETCHABILITY.

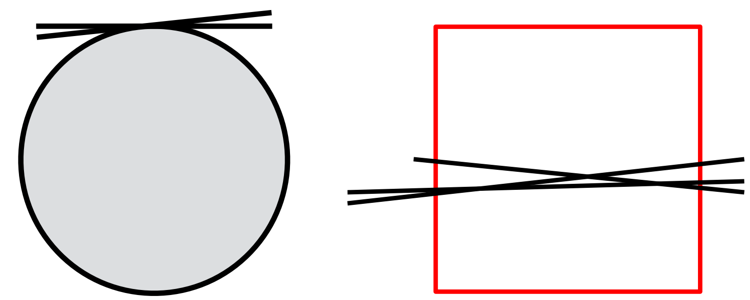
$\mathscr{A}$

$H = (V, E)$

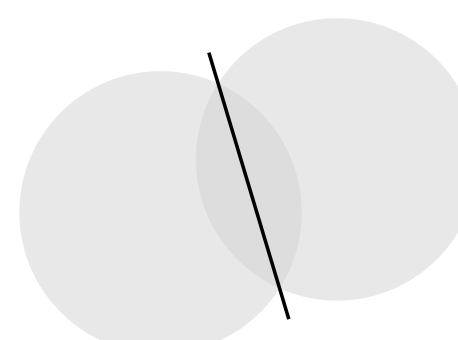




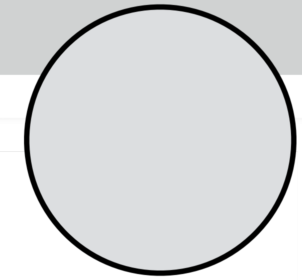
- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable



Unit disks



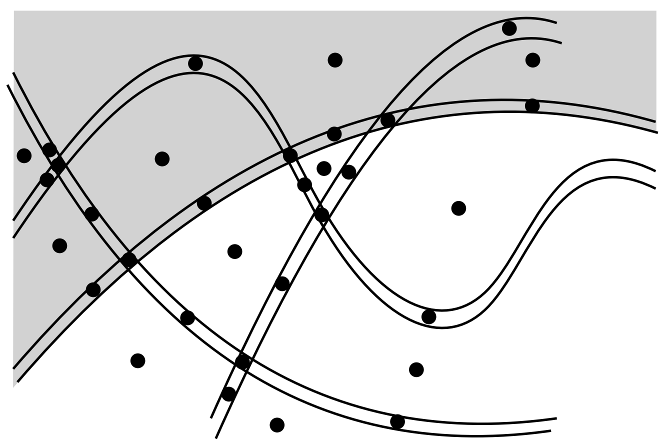
Theorem

For the family $\mathscr{F}$ of unit disks, Recognition($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

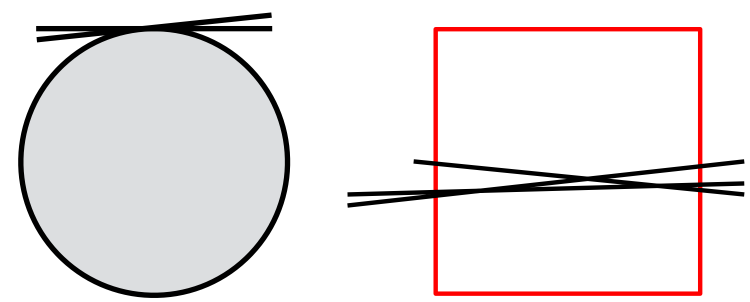
Proof: Reduction form Stretchability.

$\mathscr{A}$

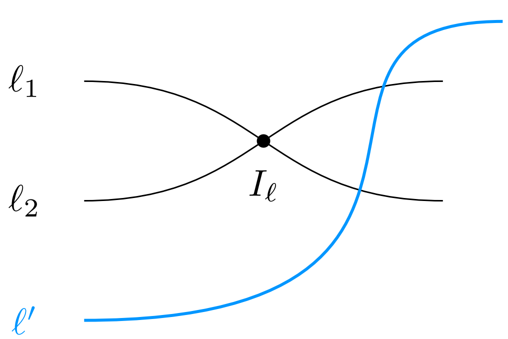
$H = (V, E)$

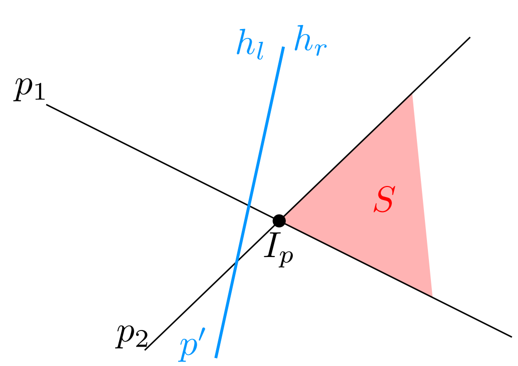


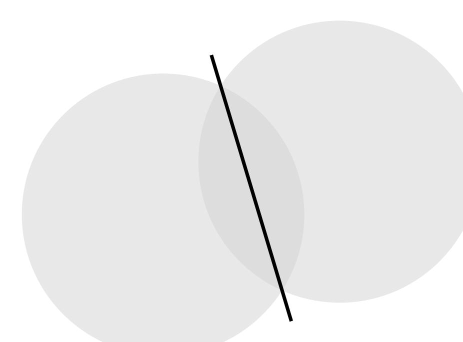
- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation

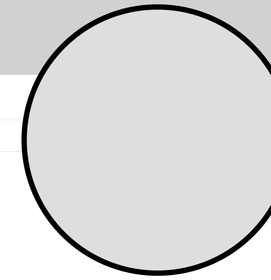


- $\mathscr{F}$-representation of $H \Longrightarrow$ $\mathscr{A}$ stretchable









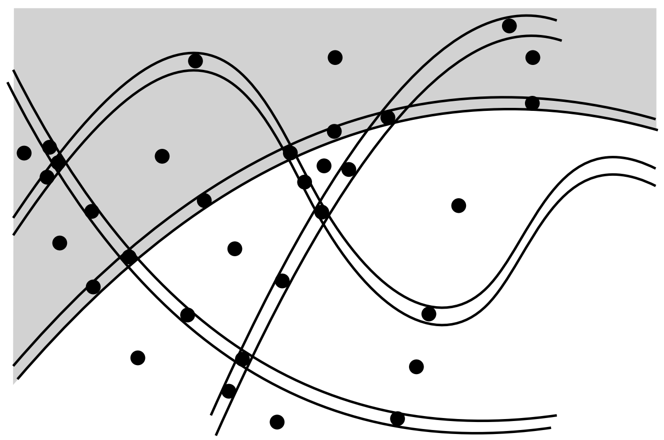
Theorem

For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

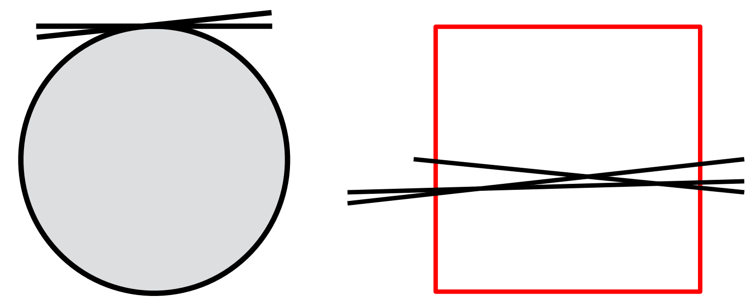
Proof: Reduction form STRETCHABILITY.

$\mathscr{A}$

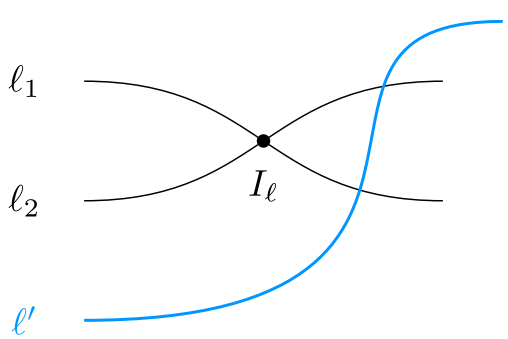
$H = (V, E)$

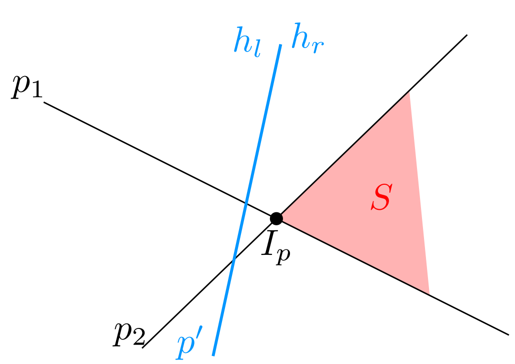


- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation

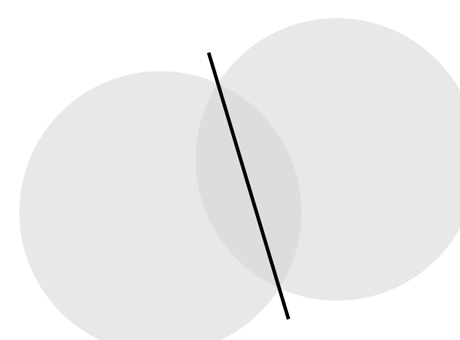


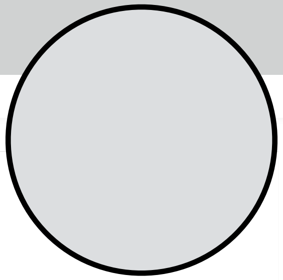
bi-curved





- $\mathscr{F}$-representation of H $\Longrightarrow$ $\mathscr{A}$ stretchable





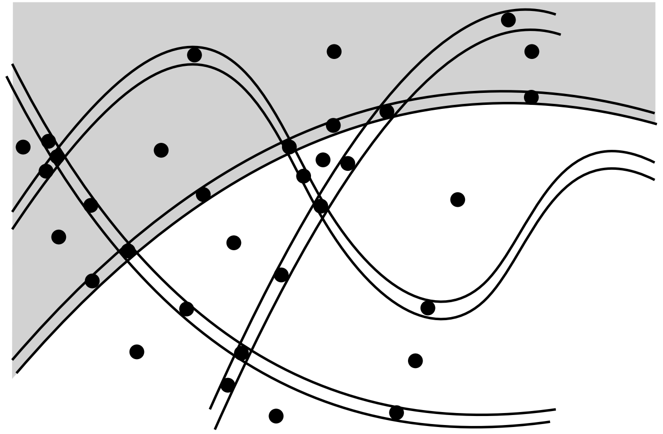
Theorem

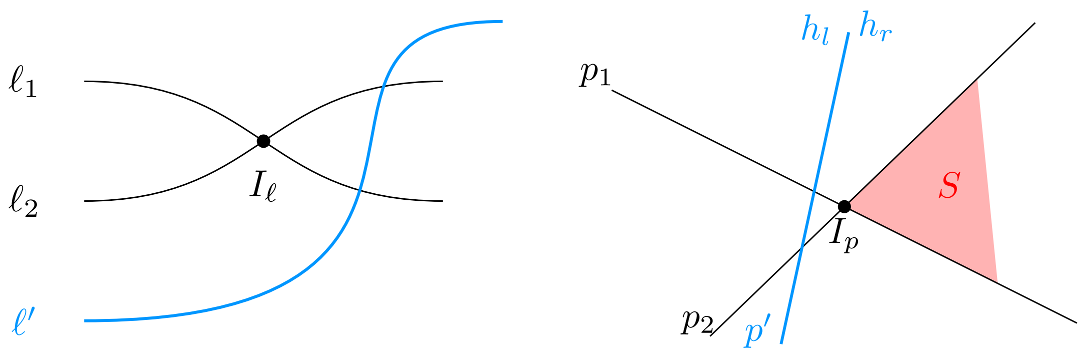
For the family $\mathscr{F}$ of unit disks, RECOGNITION($\mathscr{F}$) is $\exists\mathbb{R}$-complete.

Proof: Reduction form STRETCHABILITY.

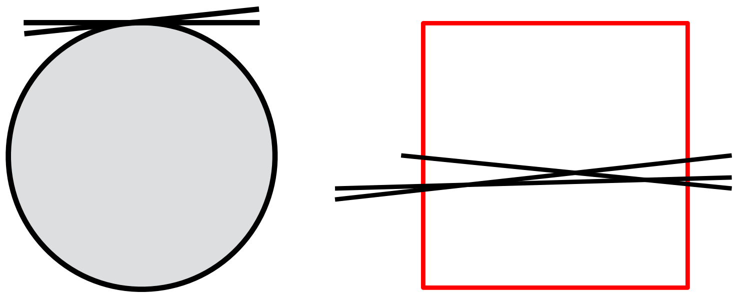
$\mathscr{A}$

$H = (V, E)$



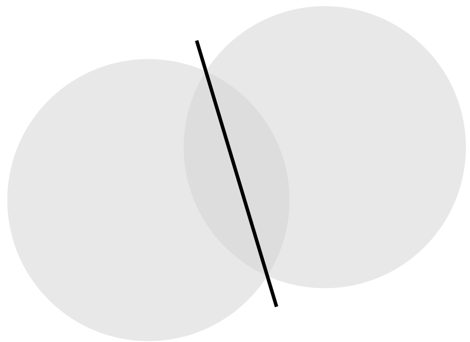


- $\mathscr{A}$ stretchable $\Longrightarrow$ H has $\mathscr{F}$-representation



bi-curved

- $\mathscr{F}$-representation of $H \Longrightarrow$ $\mathscr{A}$ stretchable



difference-separable

Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

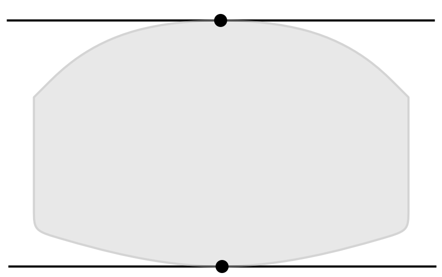
Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

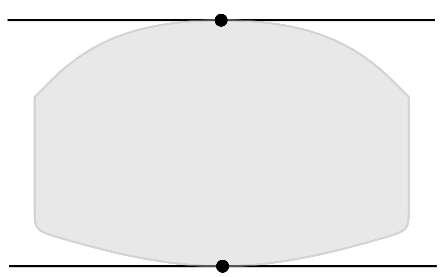
Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



Examples

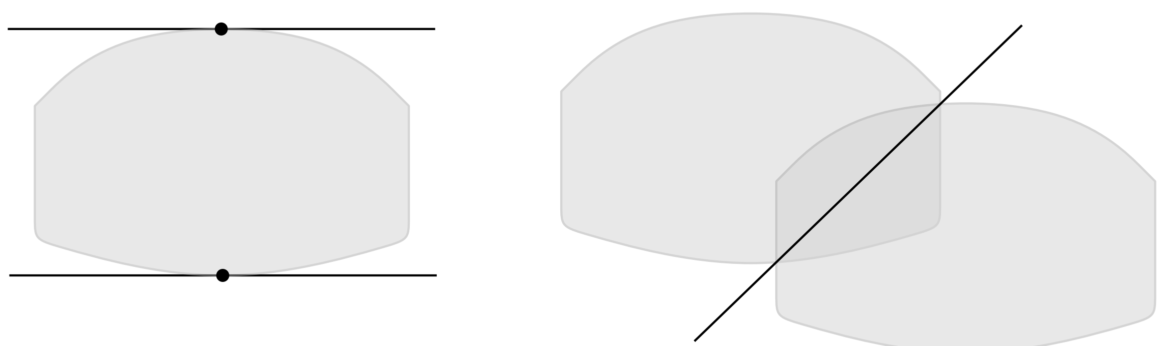
strictly convex

Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



Examples

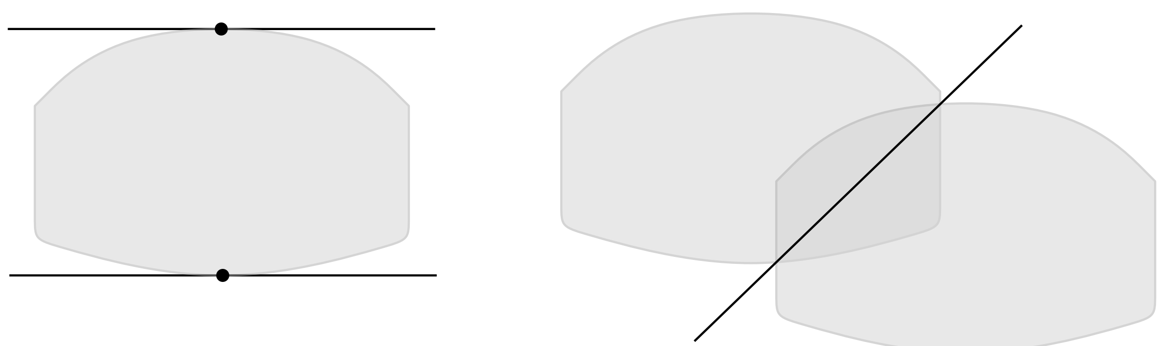
strictly convex

Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



Examples

strictly convex   convex in $\mathbb{R}^2$

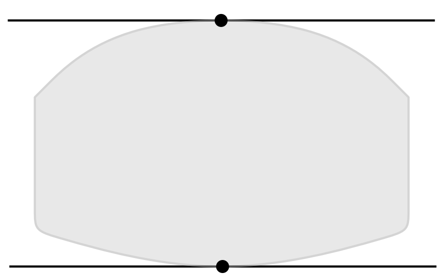
Result

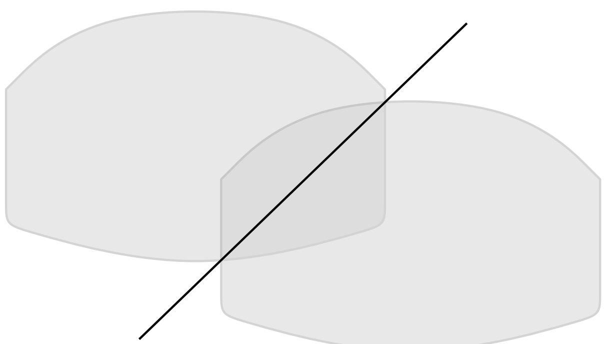
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.

Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

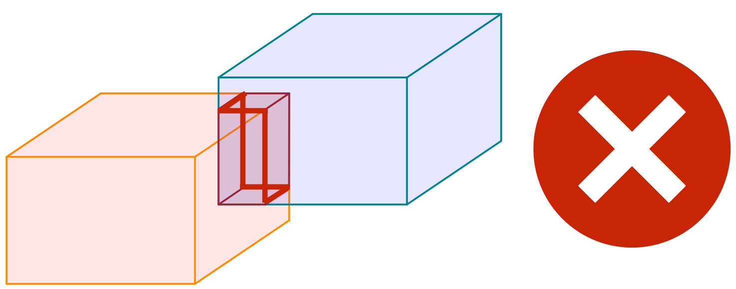




Examples

strictly convex

convex in $\mathbb{R}^2$

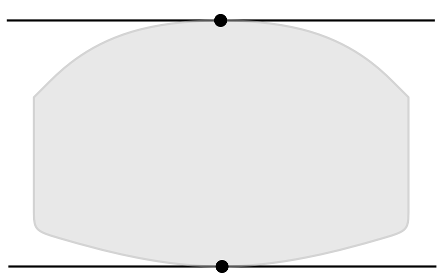


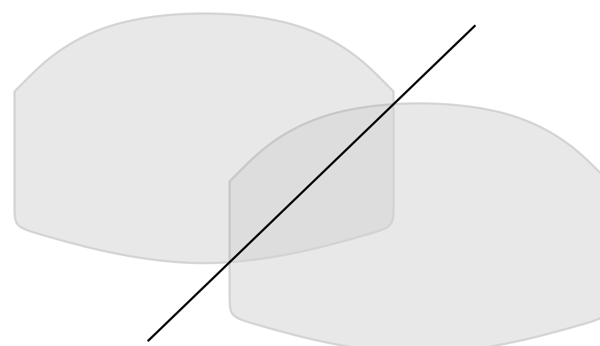
Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

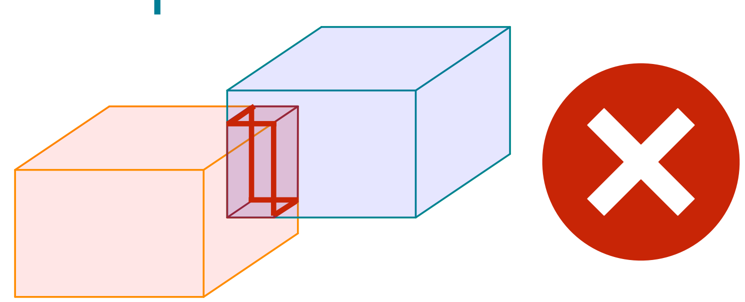




Examples

strictly convex

convex in $\mathbb{R}^2$
ellipsoid in $\mathbb{R}^d$



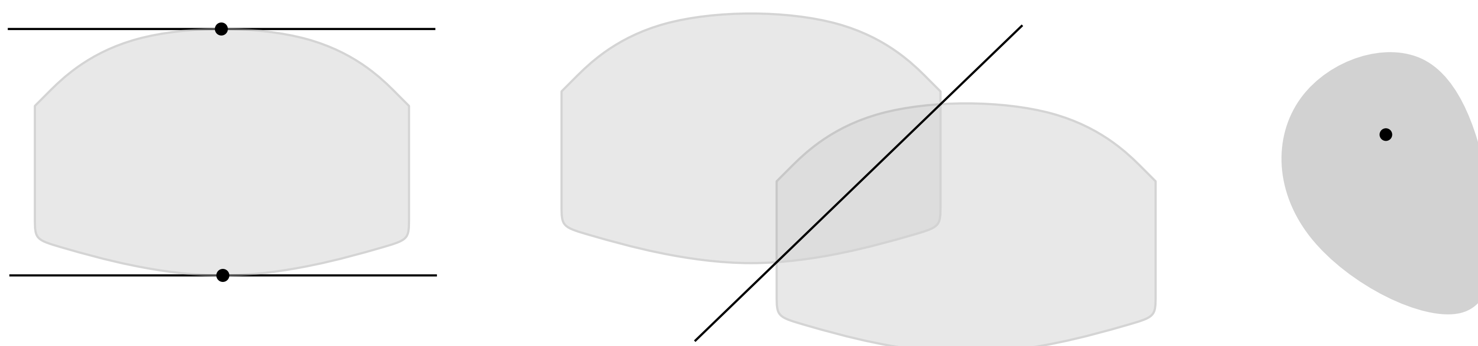
Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.

Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

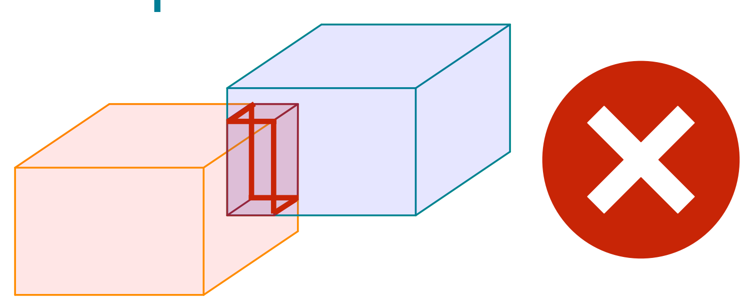


Examples

strictly convex

convex in $\mathbb{R}^2$

ellipsoid in $\mathbb{R}^d$

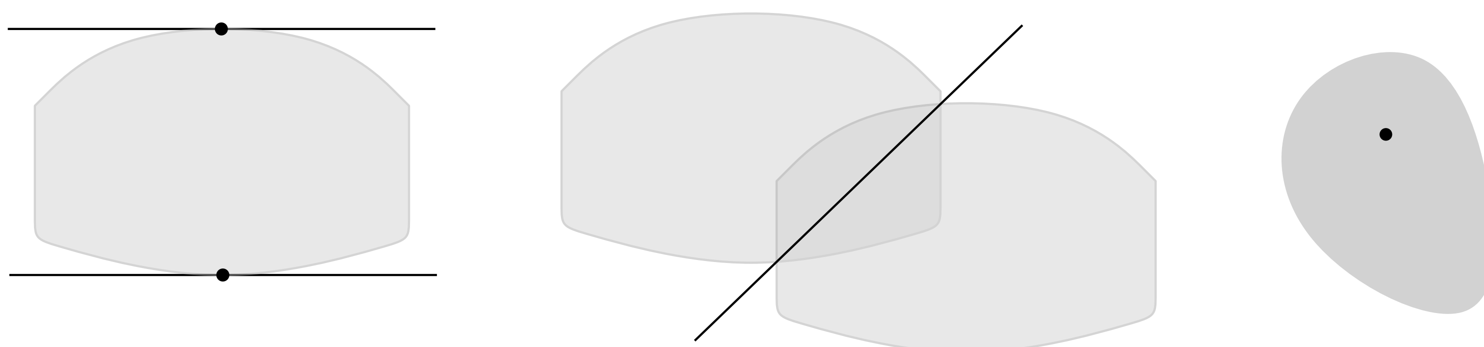


Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

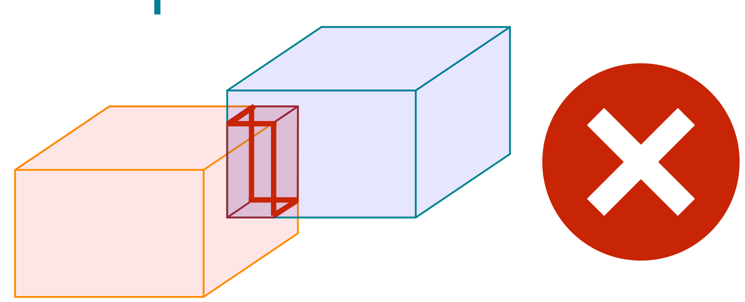
Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



Examples

strictly convex

convex in $\mathbb{R}^2$
ellipsoid in $\mathbb{R}^d$



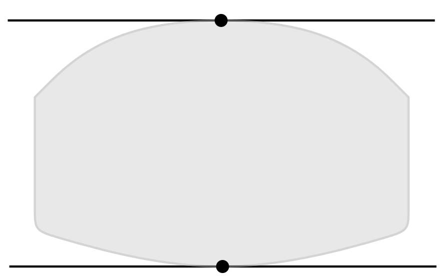
Result

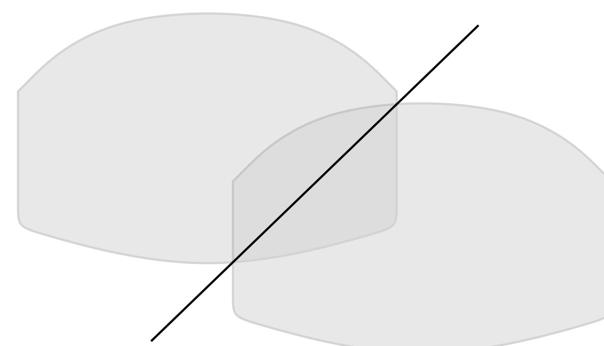
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.

Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.





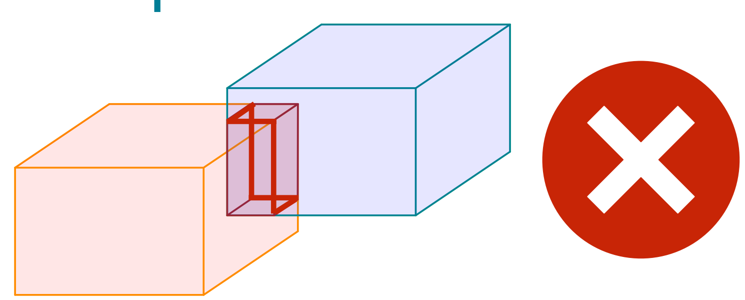
difference-separable $\implies$ convex

Examples

strictly convex

convex in $\mathbb{R}^2$

ellipsoid in $\mathbb{R}^d$

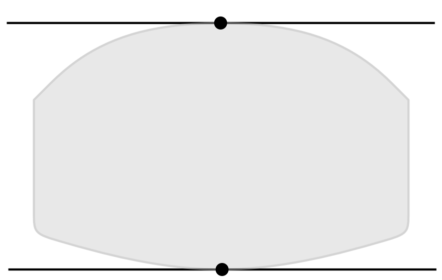


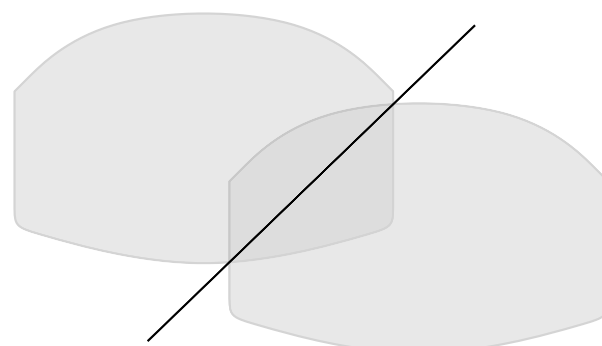
Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



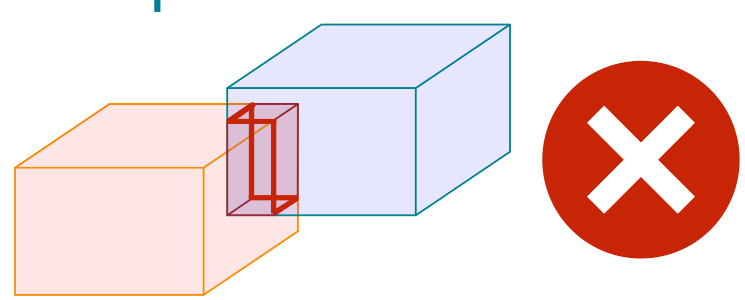


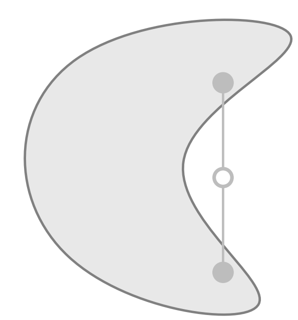
difference-separable $\implies$ convex

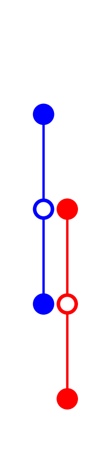
Examples

strictly convex

convex in $\mathbb{R}^2$
ellipsoid in $\mathbb{R}^d$







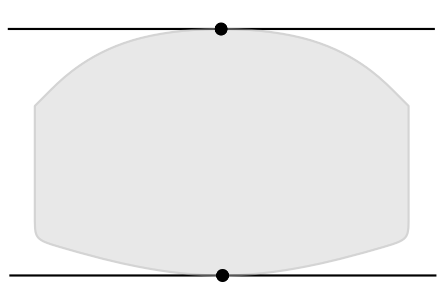
Result

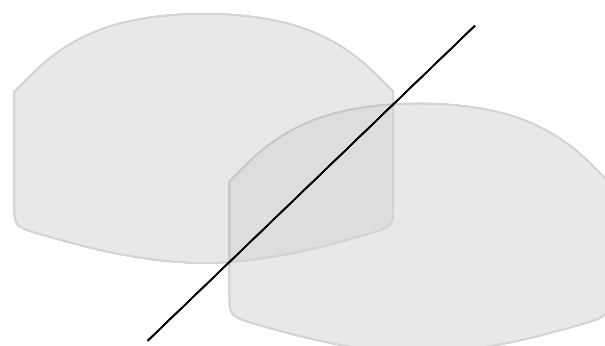
Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

Theorem

HARDNESS

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.



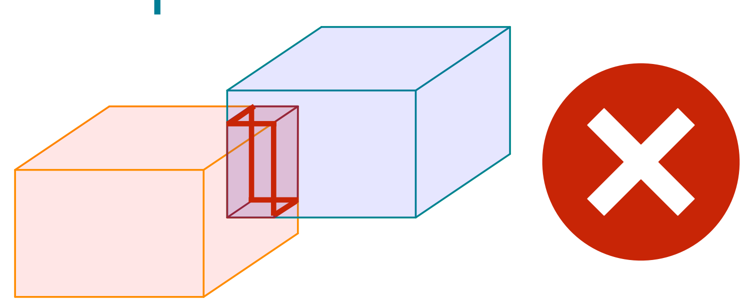


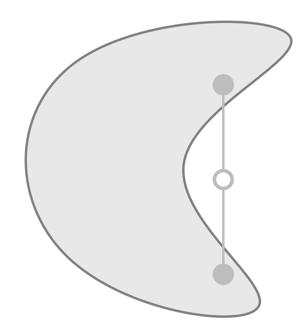
difference -separable $\Longrightarrow$ convex

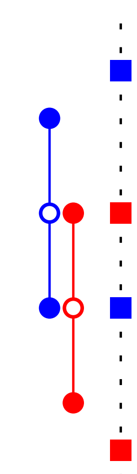
Examples

strictly convex

convex in $\mathbb{R}^2$
ellipsoid in $\mathbb{R}^d$





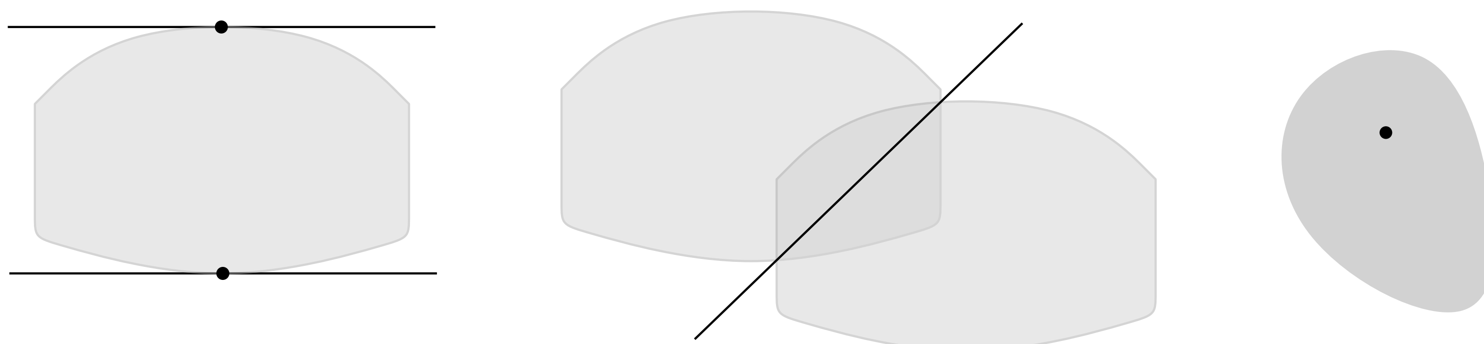


Result

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

HARDNESS | **Theorem** | MEMBERSHIP

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

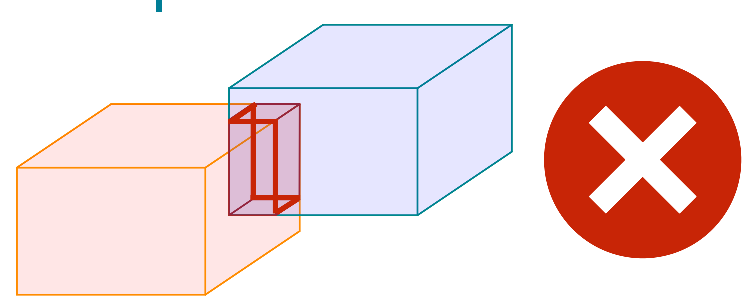


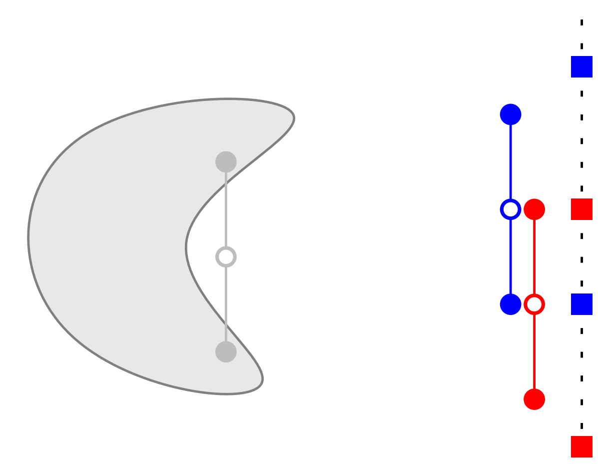
difference-separable $\Longrightarrow$ convex

Examples

strictly convex

convex in $\mathbb{R}^2$
ellipsoid in $\mathbb{R}^d$





Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.

Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ … in P
- polygon? … in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Corollary

RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-hard if C is

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

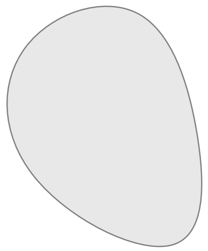
Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Corollary

RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-hard if C is

- a strictly convex set in $\mathbb{R}^2$, or



Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

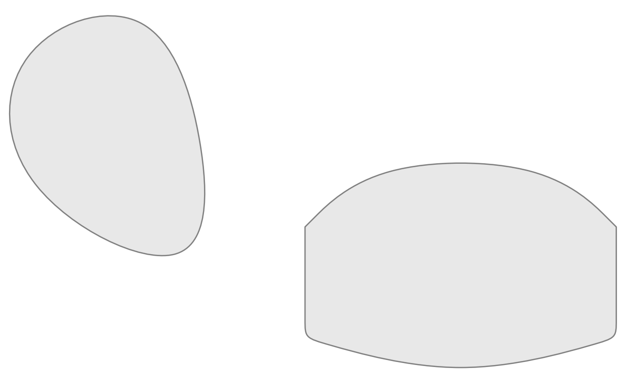
Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Corollary

RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-hard if C is

- a strictly convex set in $\mathbb{R}^2$, or
- a bi-curved convex set in $\mathbb{R}^2$, or



Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

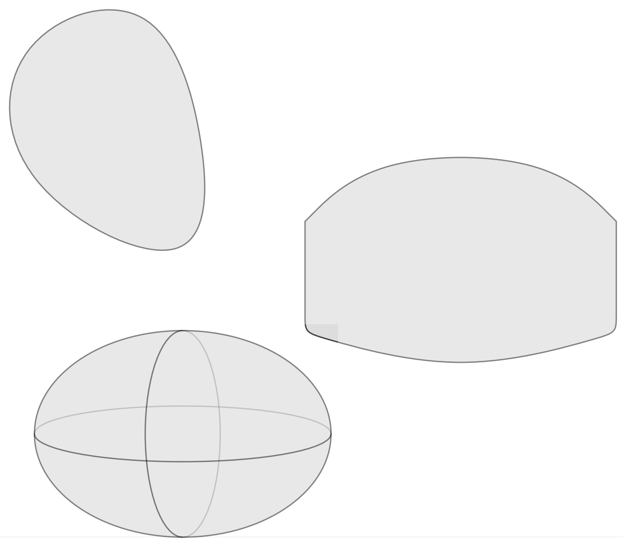
Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Corollary

RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-hard if C is

- a strictly convex set in $\mathbb{R}^2$, or
- a bi-curved convex set in $\mathbb{R}^2$, or
- an ellipsoid in $\mathbb{R}^d$, $d \geq 3$.



Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?
- Relaxing any of the conditions: Is some curvature sufficient?

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ … in P
- polygon? … in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then Recognition($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?
- Relaxing any of the conditions: Is some curvature sufficient?
- Complexity of Recognition($\mathcal{F}$) where

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ … in P
- polygon? … in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?
- Relaxing any of the conditions: Is some curvature sufficient?
- Complexity of RECOGNITION($\mathscr{F}$) where
  - ($\mathscr{F}$ is a family of **homothetic** sets)

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?
- Relaxing any of the conditions: Is some curvature sufficient?
- Complexity of RECOGNITION($\mathscr{F}$) where
  - ($\mathscr{F}$ is a family of **homothetic** sets)
  - $\mathscr{F}$ is a family of **translated and rotated copies** of set $C \subset \mathbb{R}^d$
    In particular, if C is a polygon?

Summary & Open Problems

Let $\mathbb{T}_C$ be the family of **translates** of set $C \subset \mathbb{R}^d$.

- $d = 1$ ... in P
- polygon? ... in NP

Theorem

Let C be a bi-curved, difference-separable, computable (convex) set.
Then RECOGNITION($\mathbb{T}_C$) is $\exists\mathbb{R}$-complete.

- Is each (bi-curved) difference-separable set in $\mathbb{R}^3, \mathbb{R}^4, \ldots$ an ellipsoid?
- Relaxing any of the conditions: Is some curvature sufficient?
- Complexity of RECOGNITION($\mathscr{F}$) where
  - ($\mathscr{F}$ is a family of **homothetic** sets)
  - $\mathscr{F}$ is a family of **translated and rotated copies** of set $C \subset \mathbb{R}^d$
    In particular, if C is a polygon?

Appendix

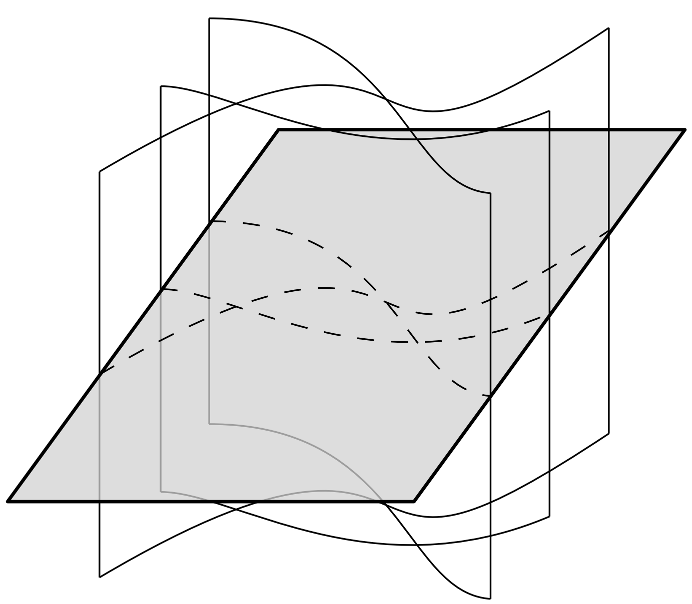
d-STRETCHABILITY

d-STRETCHABILITY

Is a given (partial) pseudohyperplane arrangement in $\mathbb{R}^d$ stretchable?

Theorem

The decision problem d-STRETCHABILITY is $\exists\mathbb{R}$-complete.



Membership

Lemma

Let C be a computable set.

Then Recognition($\mathbb{T}_C$) is contained in $\exists\mathbb{R}$.

Membership

Lemma

Let C be a computable set.

Then Recognition($\mathbb{T}_C$) is contained in $\exists\mathbb{R}$.

Proof:

Membership

Lemma

Let C be a computable set.

Then RECOGNITION($\mathbb{T}_C$) is contained in $\exists\mathbb{R}$.

Proof:

Certificate

- point coordinates
- translation vectors

Membership

Lemma

Let C be a computable set.

Then RECOGNITION($\mathbb{T}_C$) is contained in $\exists\mathbb{R}$.

Proof:

Certificate

- point coordinates
- translation vectors

Since C is computable, a verification algorithm can check such an certificate efficiently on real RAM.

Homothets

Homothets

Theorem

Let C be a bi-curved, convex set in $\mathbb{R}^2$ and $\mathbb{H}_C$ be the family of homothets of set C. Then RECOGNITION($\mathbb{H}_C$) is $\exists\mathbb{R}$-hard.