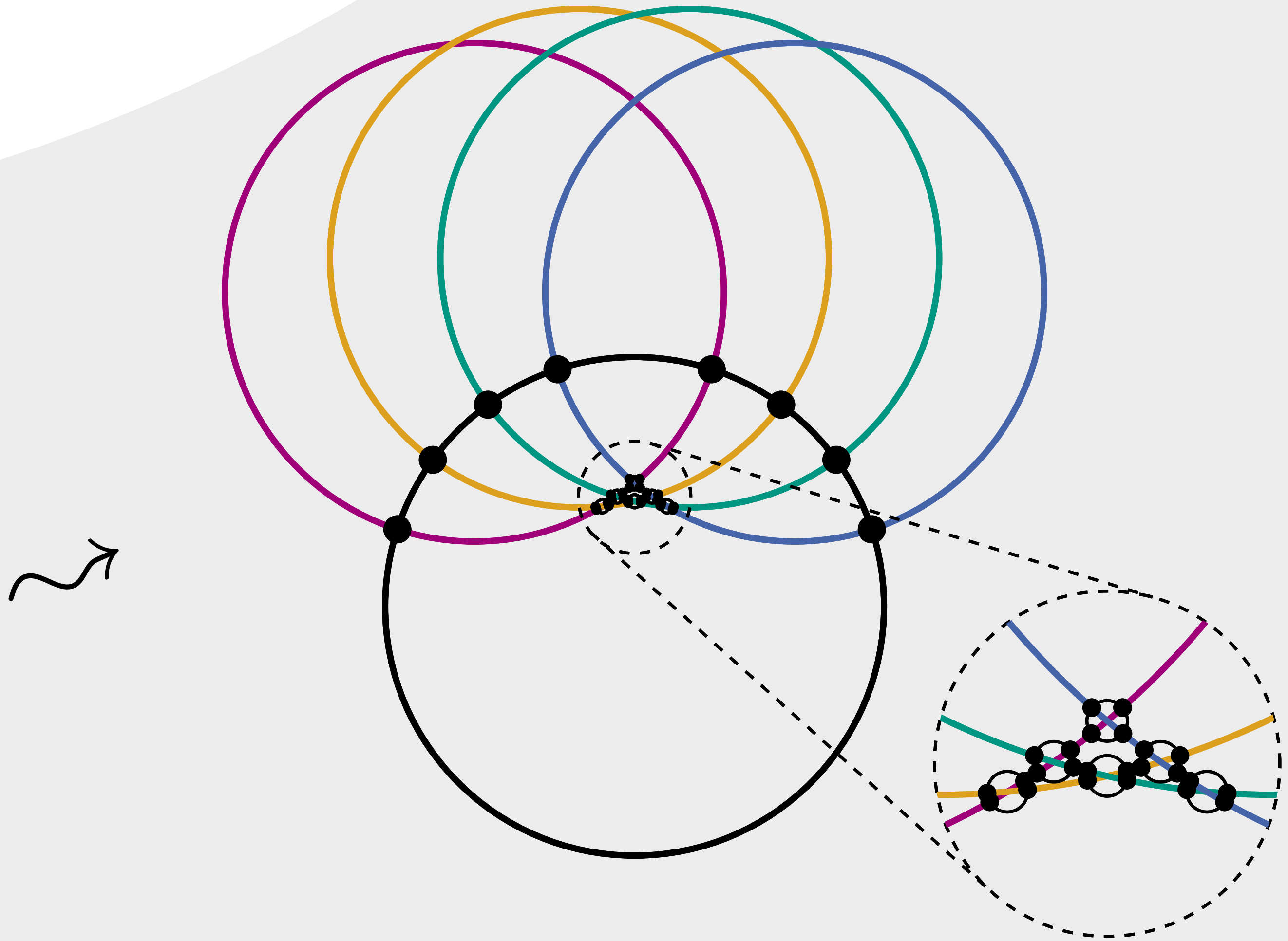
On the Complexity of Lombardi Graph Drawing

GD 2023 · 20.9.2023

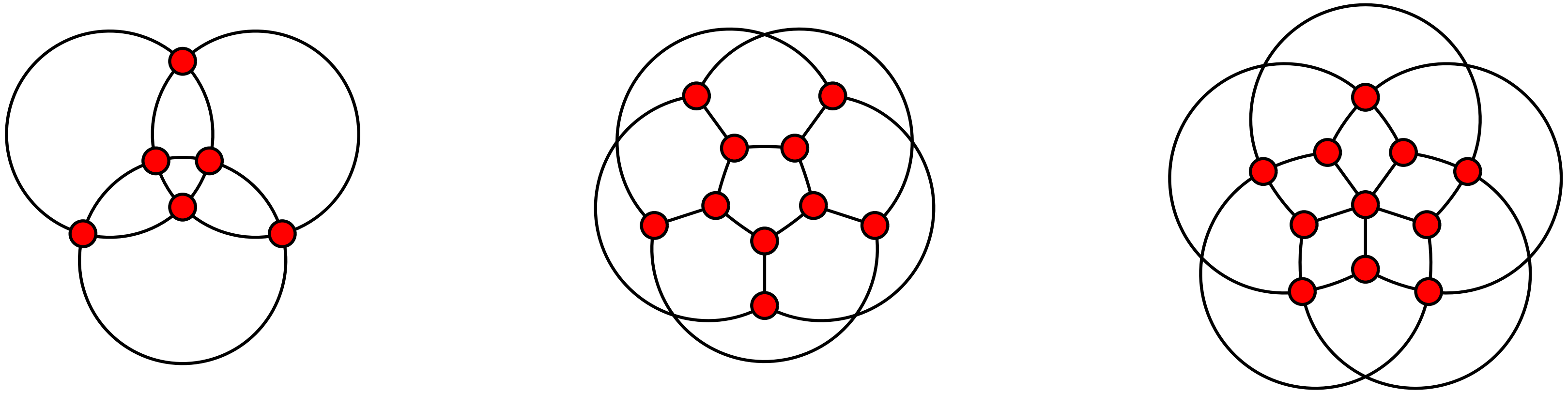
Paul Jungeblut



Lombardi Drawing

Definition: Lombardi Drawing

Vertices	points in $\mathbb{R}^2$
Edges	circular arcs (or line segments)
Constraint	perfect angular resolution



Images created with the Lombardi Spirograph by David Eppstein.

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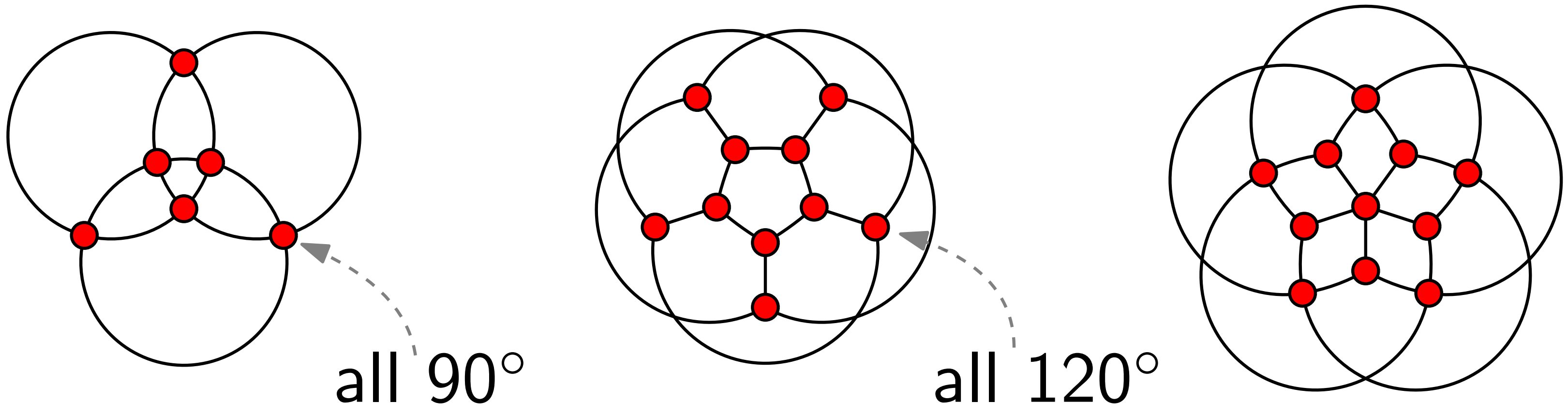

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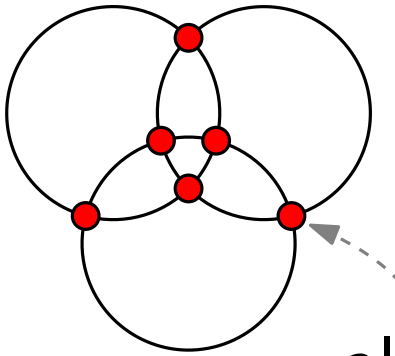
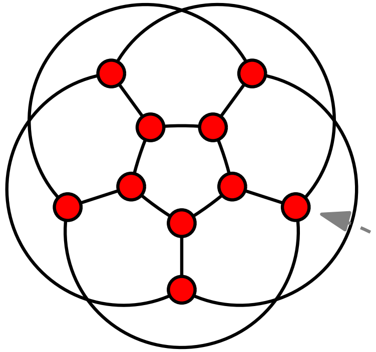
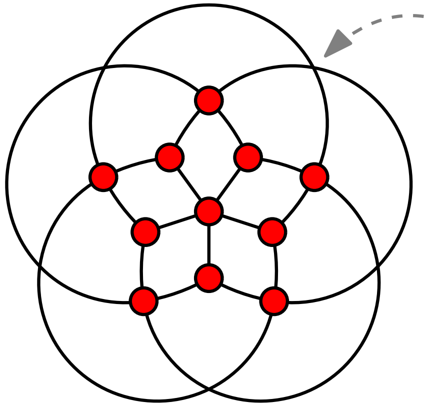


all 90°



all 120°



may contain crossings

(arbitrary crossing angles)

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Our Result

Input: graph G
rotation system R

Question: Does G have a Lombardi drawing respecting R?

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It is $\exists\mathbb{R}$-complete to decide whether G has a Lombardi drawing respecting R.

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$$

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Complexity Class $\exists\mathbb{R}$

- appears frequently in computational geometry and graph drawing
- difficulty = solving polynomial system of equations and inequalities
- Formally: reducible to

$$\exists X_1, \dots X_n \in \mathbb{R} : \varphi(X_1, \dots, X_n)$$

real valued variables

polynomial equations and inequalities

Related Work

Lombardi Graph Drawing

GD 2010 introduced by Duncan, Eppstein, Goodrich, Kobourov and Nöllenburg

Always exist for 2-degenerate, trees, cacti, subcubic, outerpaths, . . .

Variants planar, circular, . . .

Complexity no general results yet

Related Work

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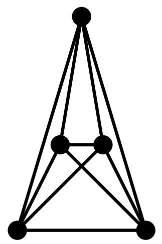
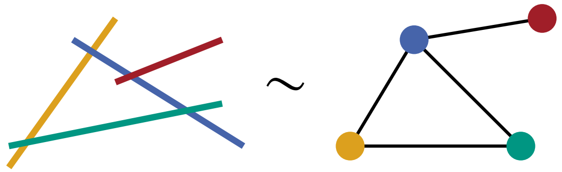
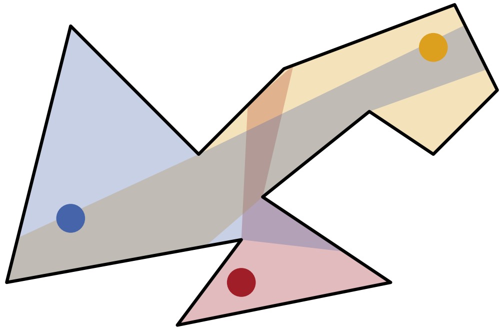


RAC-drawing

recognition of intersection graphs





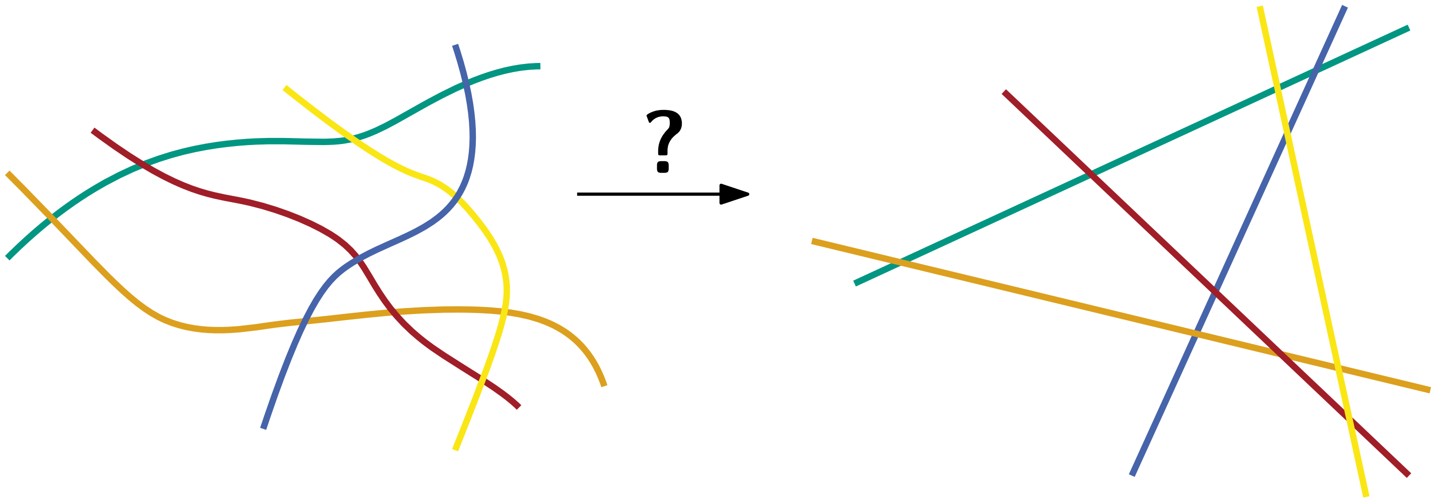
art gallery problem

⋮

Stretchability

Input: pseudolines

Want: lines in $\mathbb{R}^2$



Simple Stretchability:

- every two pseudolines intersect exactly once
- no triple intersections

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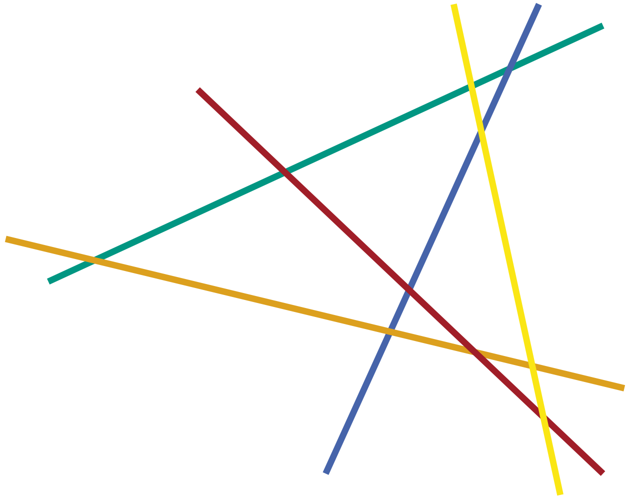

Theorem:
(Mnëv 1988)
Simple Stretchability is $\exists\mathbb{R}$-complete in $\mathbb{R}^2$.

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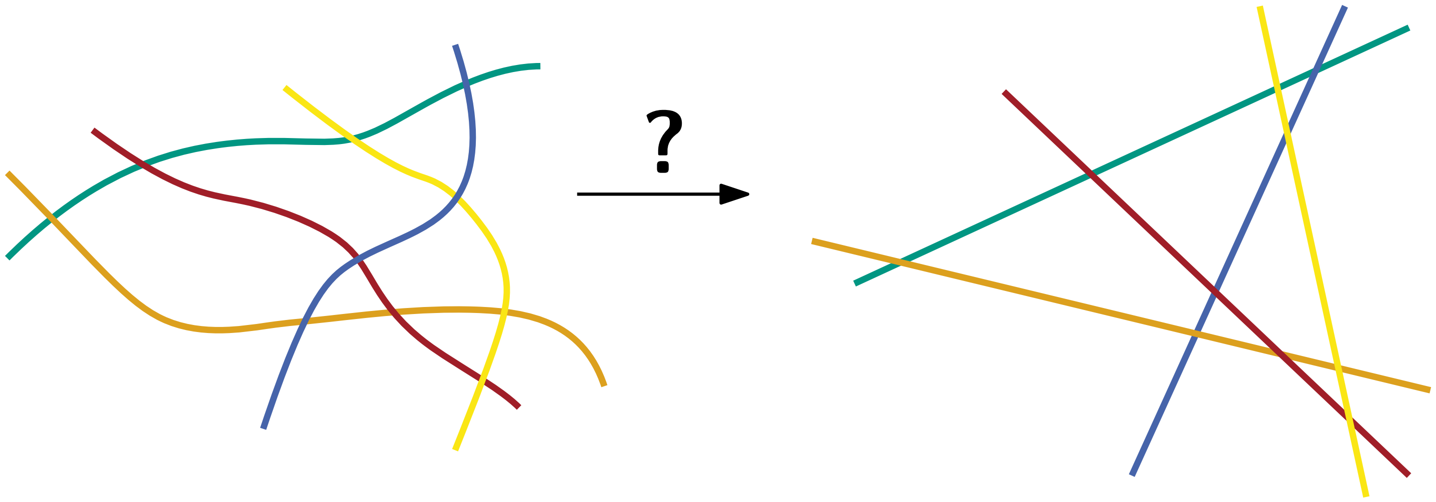

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Observation:
(Bieker, Bläsius, Dohse, Jungeblut 2023)
Simple Stretchability is $\exists\mathbb{R}$-complete in $\mathbb{H}^2$.

$\mathbb{H}^2$ – Hyperbolic Plane

- non-Euclidean geometry
- has already been used in the literature to construct Lombardi drawings

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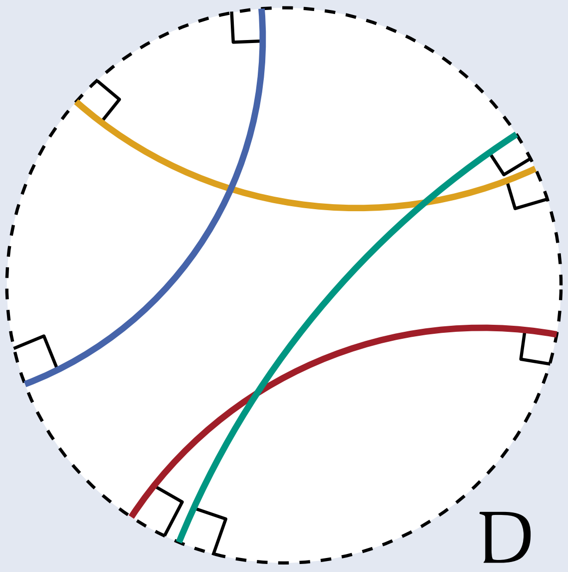
Poincaré Disk Model

- embeds $\mathbb{H}^2$ into $\mathbb{R}^2$
- $\mathbb{H}^2$ is mapped to the interior of a unit disk D
- hyperbolic lines $\rightsquigarrow$ circular arcs orthognal to D
- conformal: preserves angles



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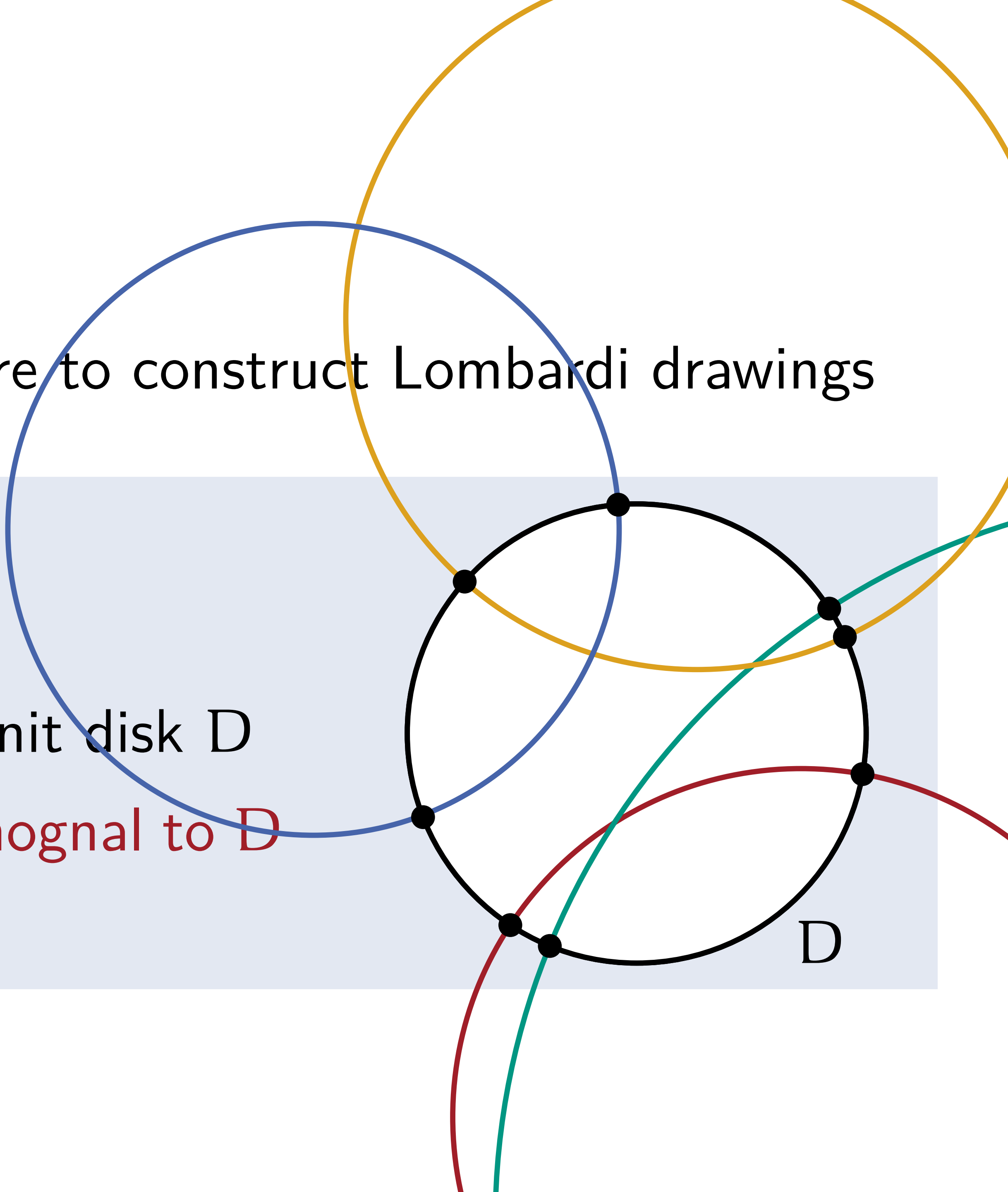
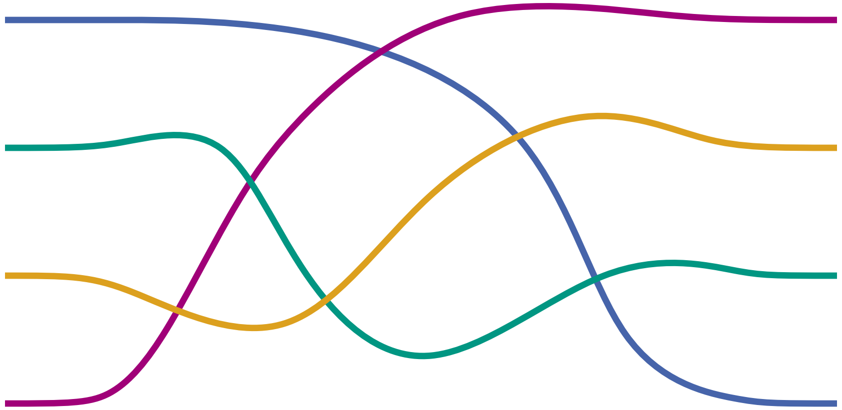


Reduction (Sketch)



1) pseudoline arrangement A

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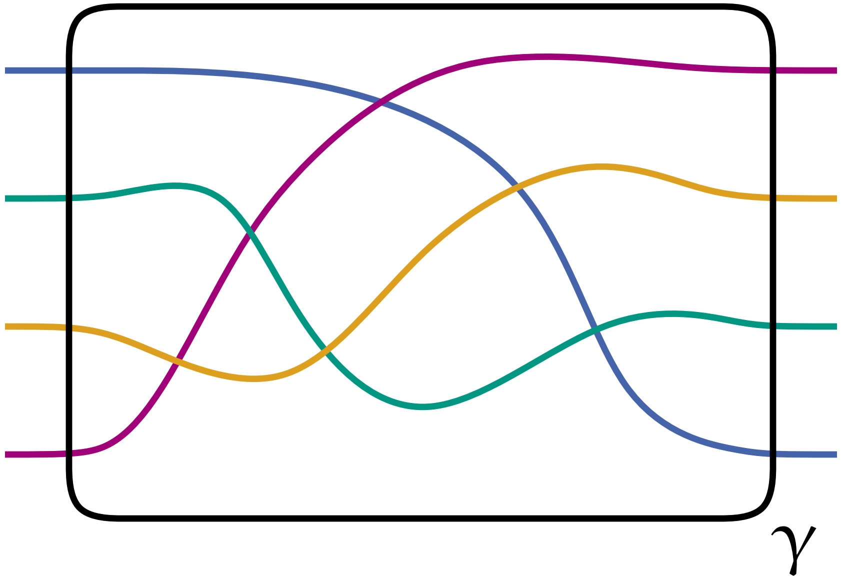

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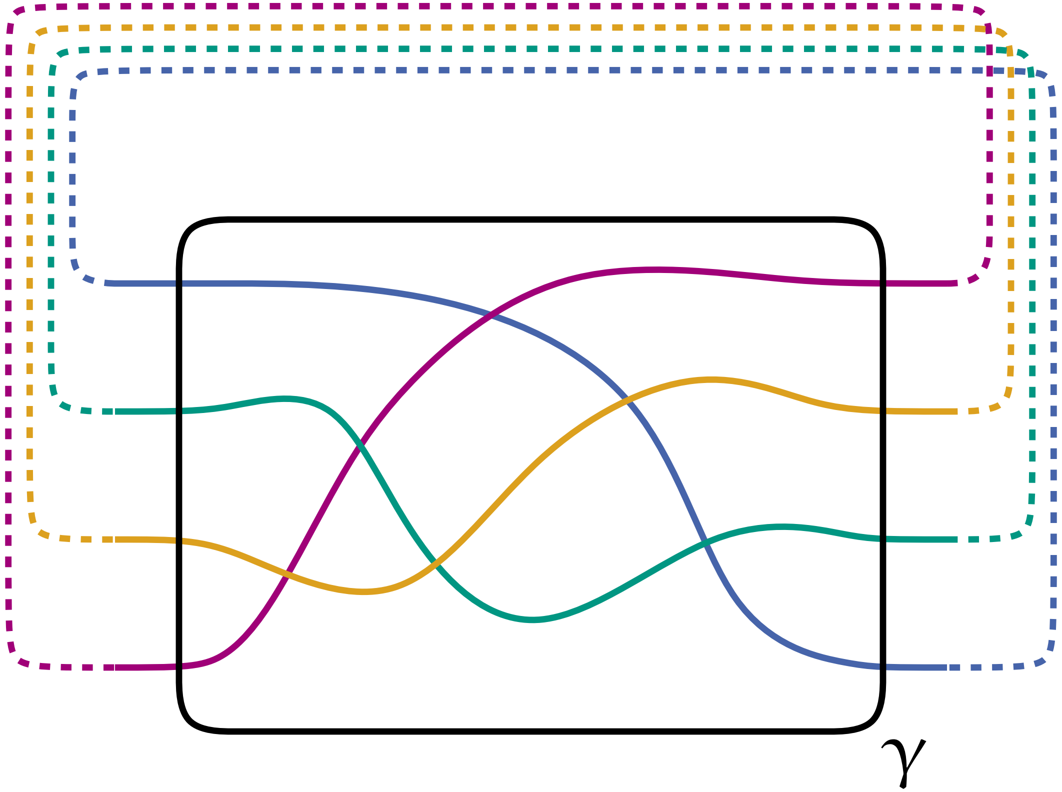

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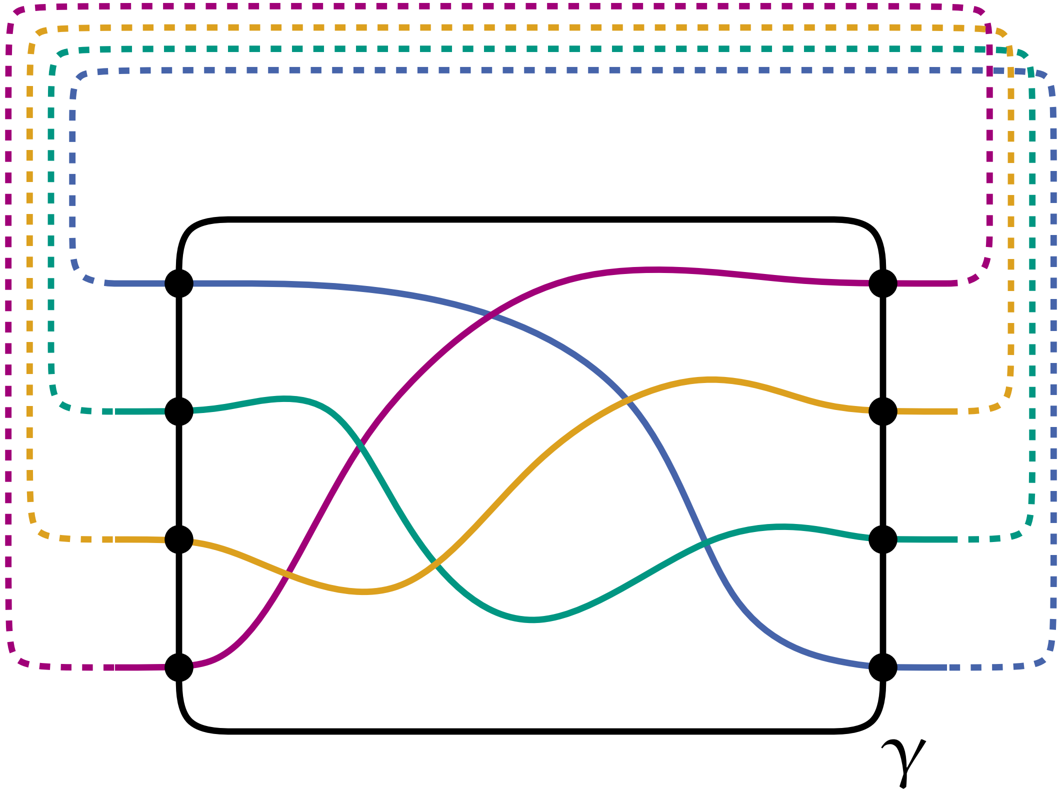

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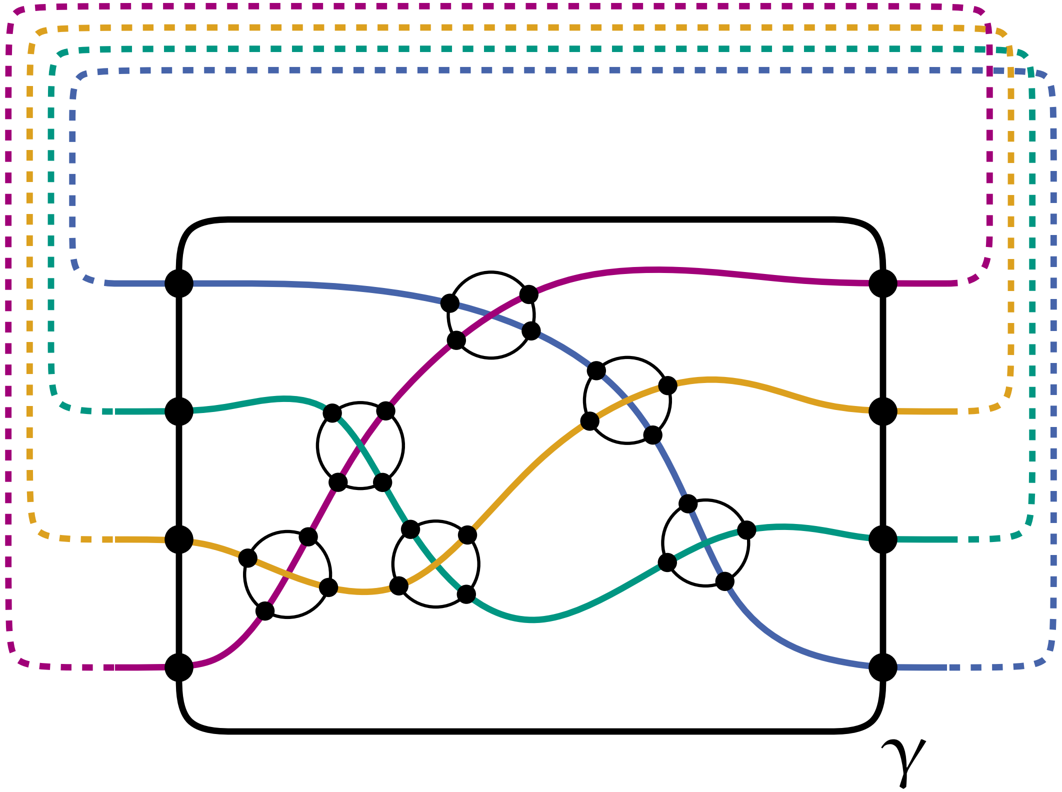

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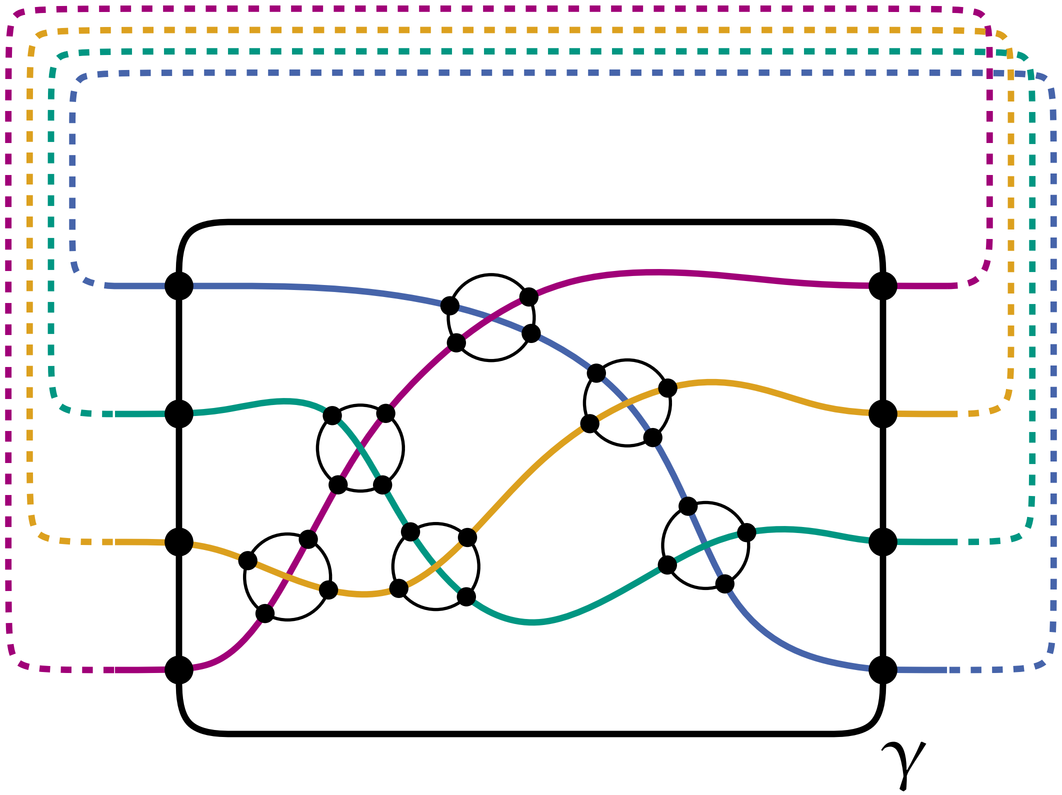

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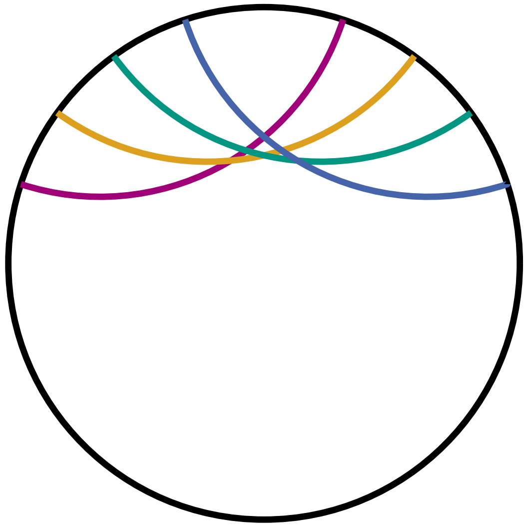
$\rightsquigarrow$ Lombardi instance G

Stretchable $\rightsquigarrow$ Lombardi

Recall:
A is stretchable in $\mathbb{R}^2$
$\iff$
A is stretchable in $\mathbb{H}^2$

Construct Lombardi Drawing:

- take realization of A in the Poincaré disk



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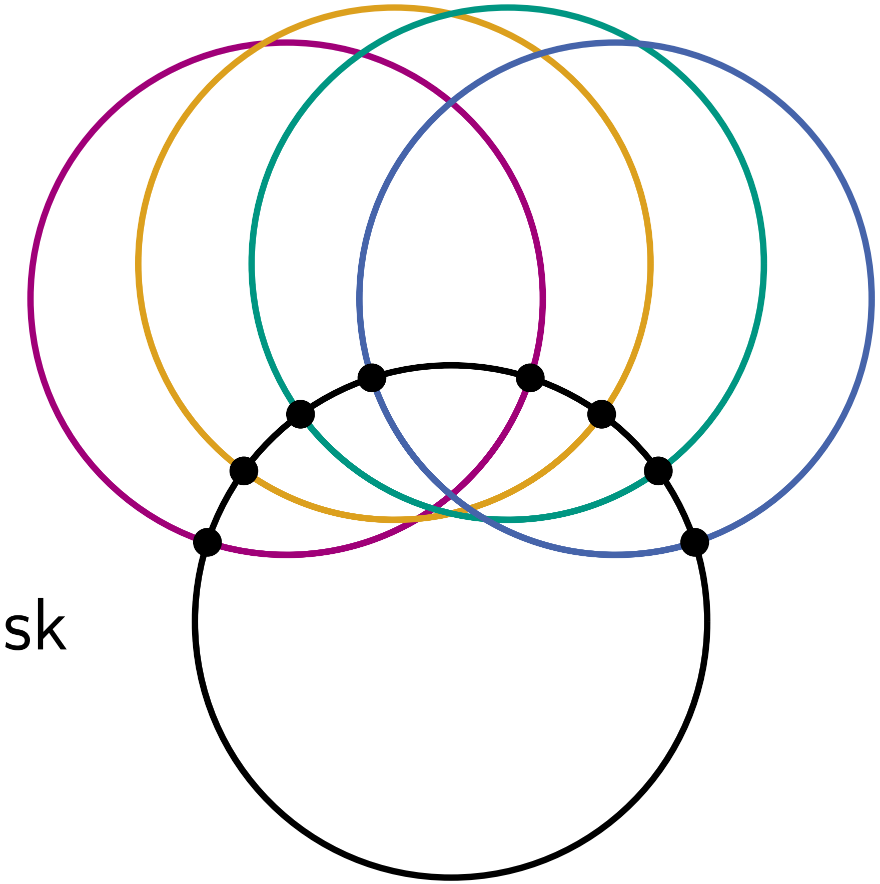

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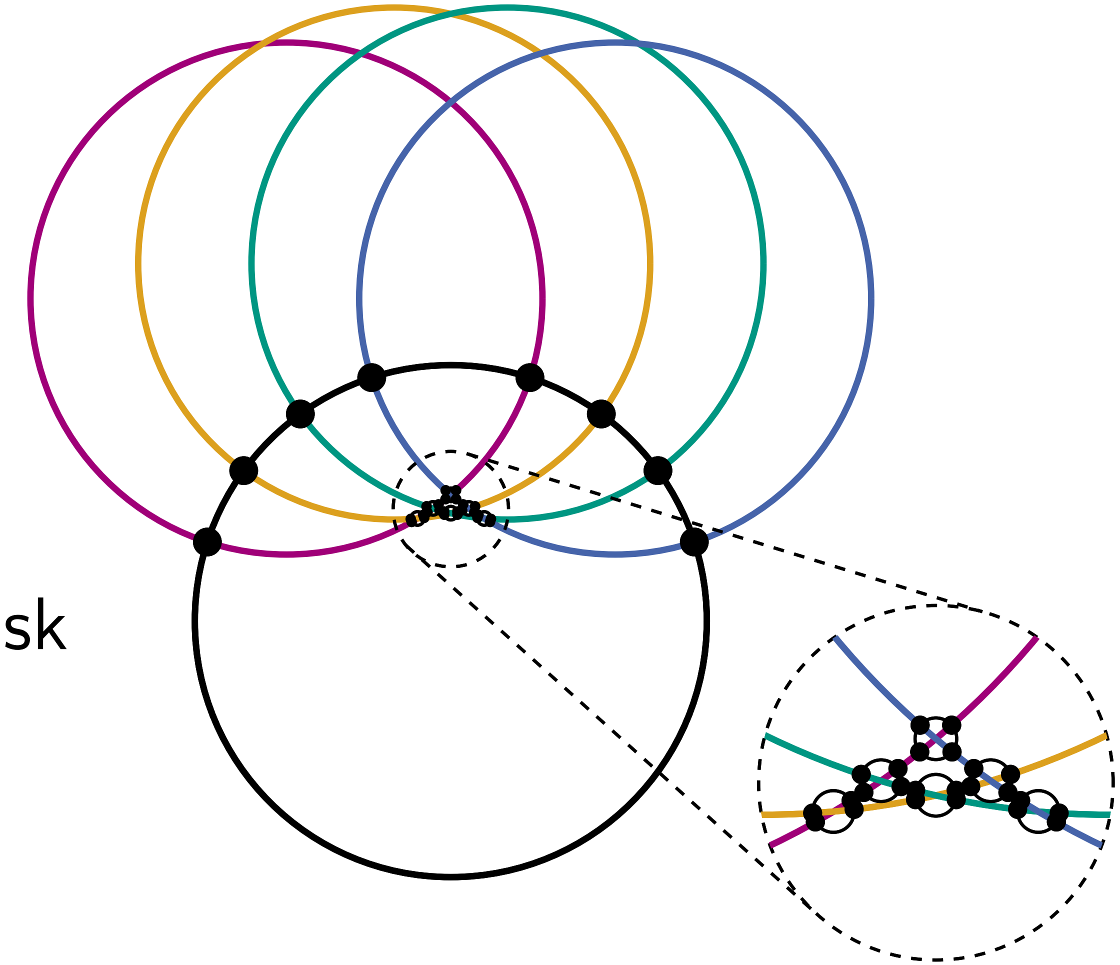
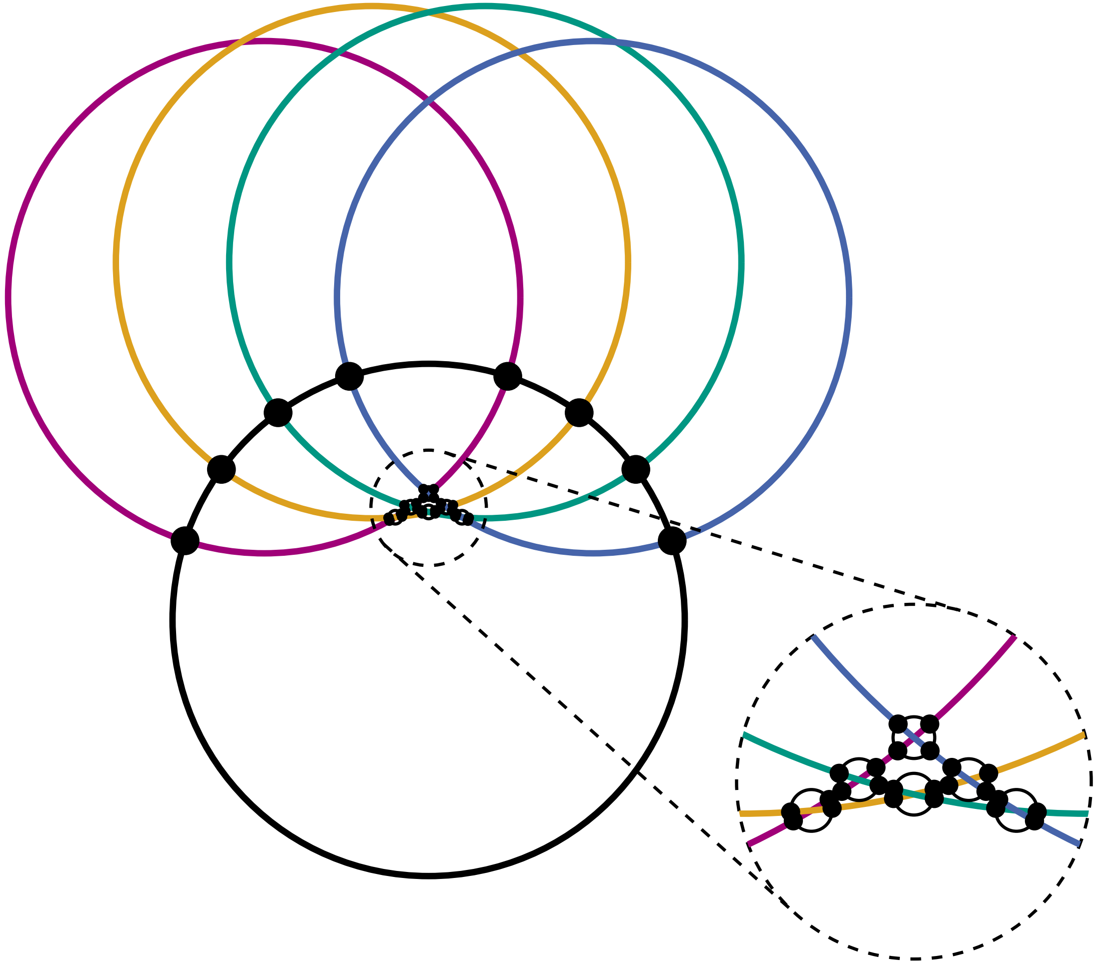


Construct Lombardi Drawing:

- take realization of A in the Poincaré disk
- extend circular arcs to circles
- add circles around intersections

Lombardi $\rightsquigarrow$ Stretchable

Problem: A Lombardi drawing of G might not look like a Poincaré disk with hyperbolic lines.



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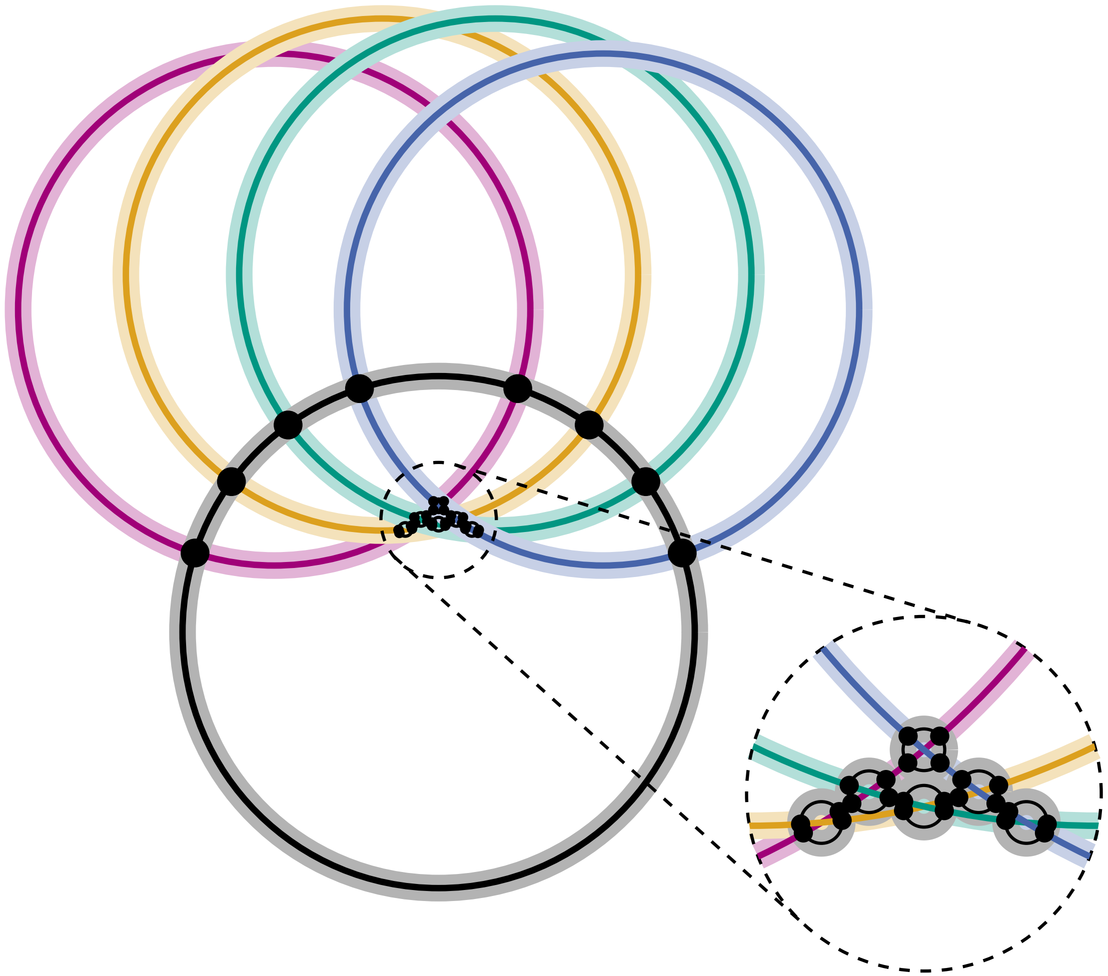
Solution: Circle gadgets that force cycles in G to be drawn as circles.



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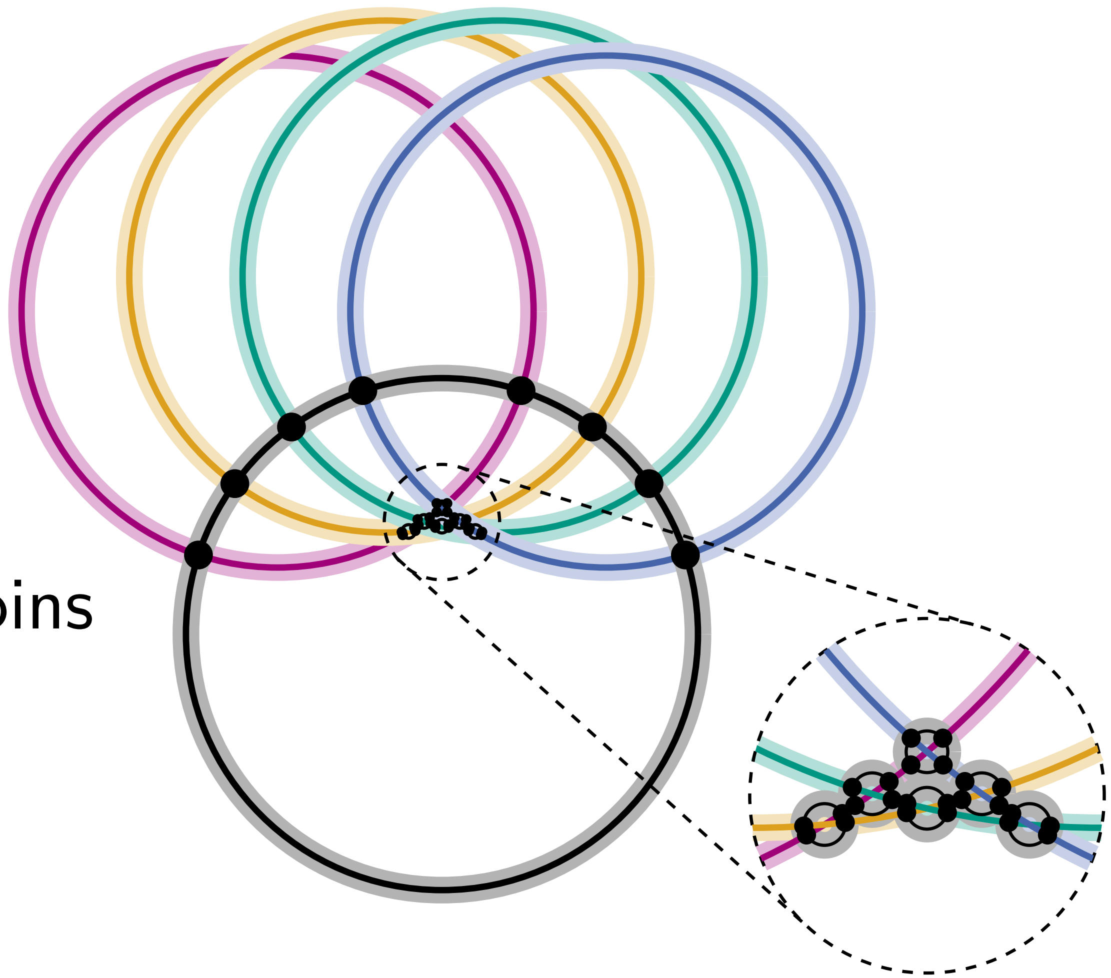


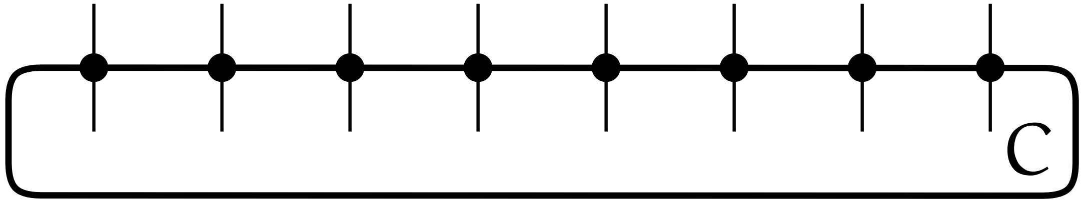
Extract line arragnement:

- interpret as Poincaré disk
- little circles $\rightsquigarrow$ same order of intersectoins

$\Rightarrow$ A is stretchable

Circle Gadget

Given: cycle C, 4-regular



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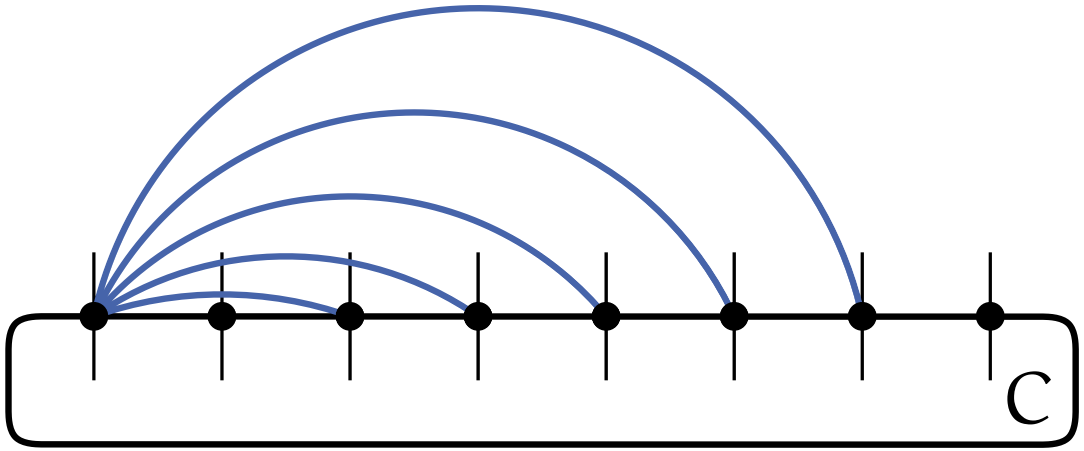

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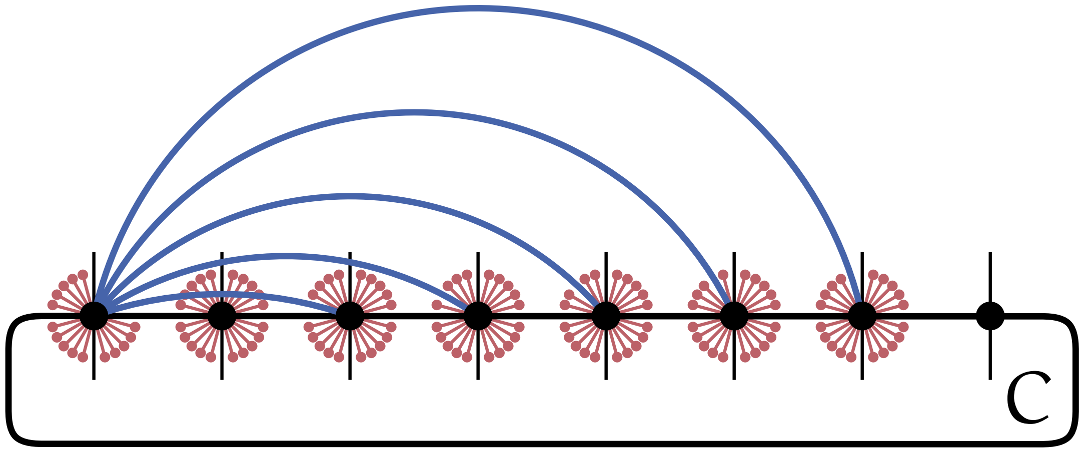

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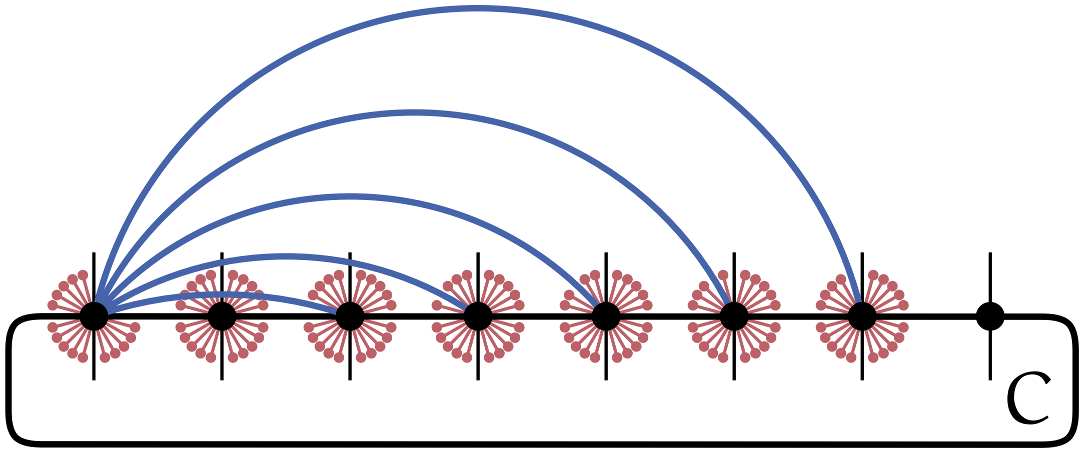

Lemma:
C must be drawn as a circle.

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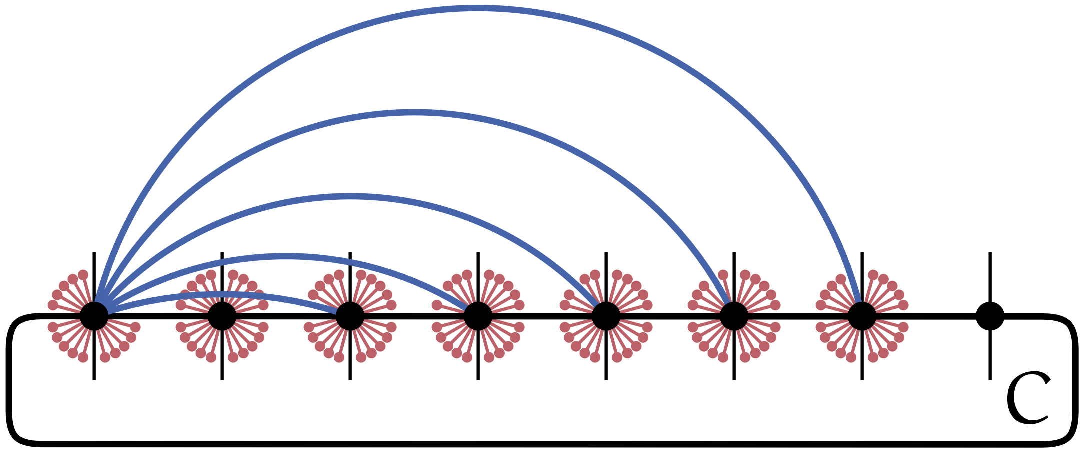

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Idea: perfect angular resolution + fixed rotation system R

$\Rightarrow$ all angles are known

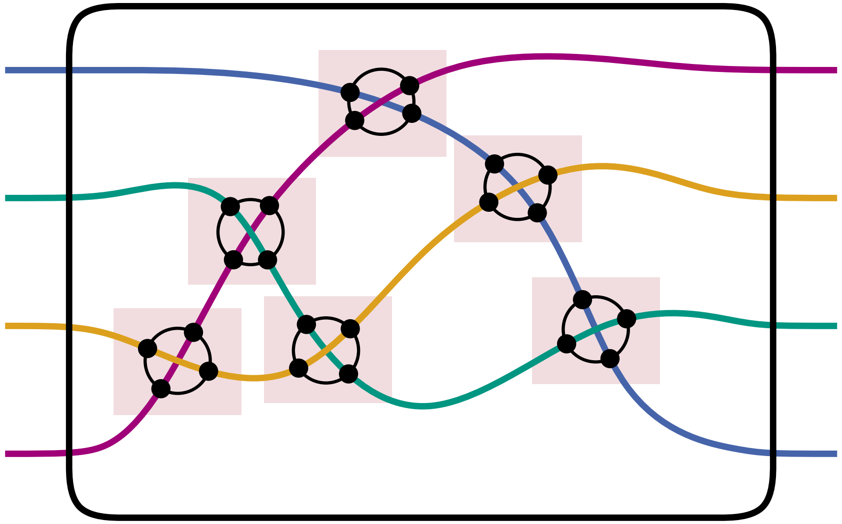
$\Rightarrow$ characterization of arc-polygons
(Eppstein, Frishberg, Osegueda 2023)

$\Rightarrow$ C must be drawn as a circle $\square$

Open Problems

Problem 1:

Planar Lombardi drawings:



additional crossings are caused by circle gadgets

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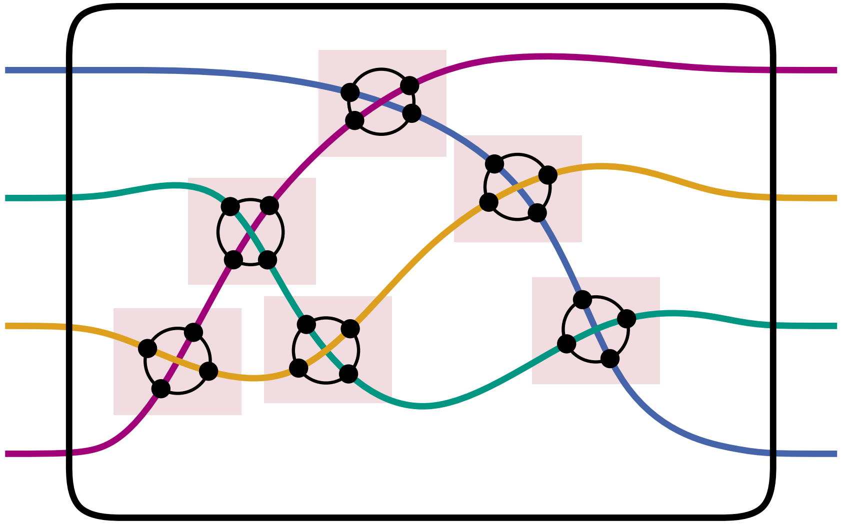

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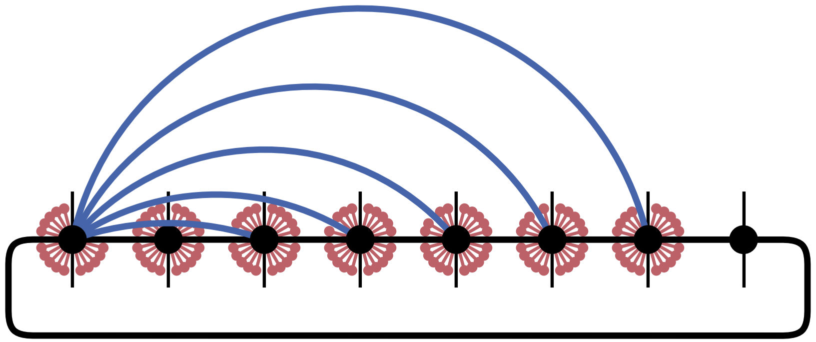
Problem 2:

Without fixed rotation system?



What are the angles between the edges?