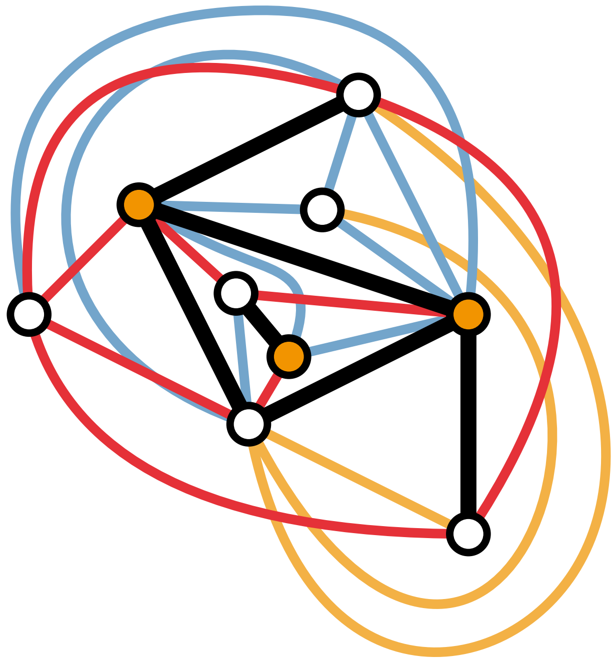
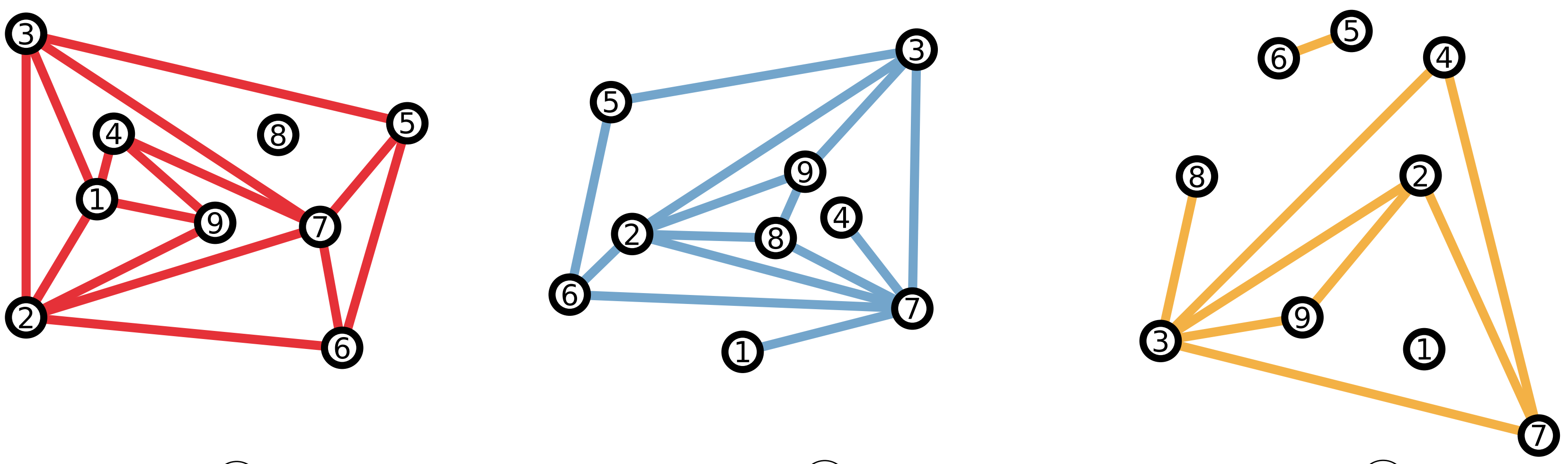


Parameterized Complexity of Simultaneous Planarity

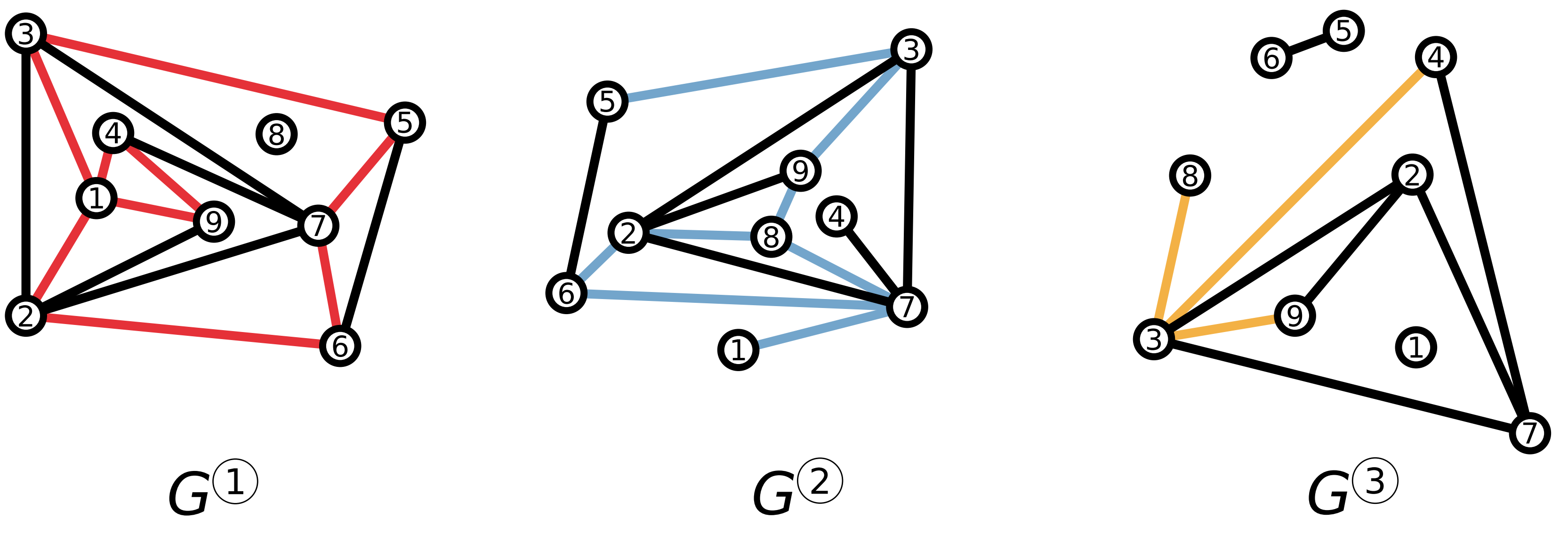
Simon D. Fink, Matthias Pfretzschner, Ignaz Rutter

Simultaneous Planarity



$G^{①}$ $G^{②}$ $G^{③}$

Simultaneous Planarity



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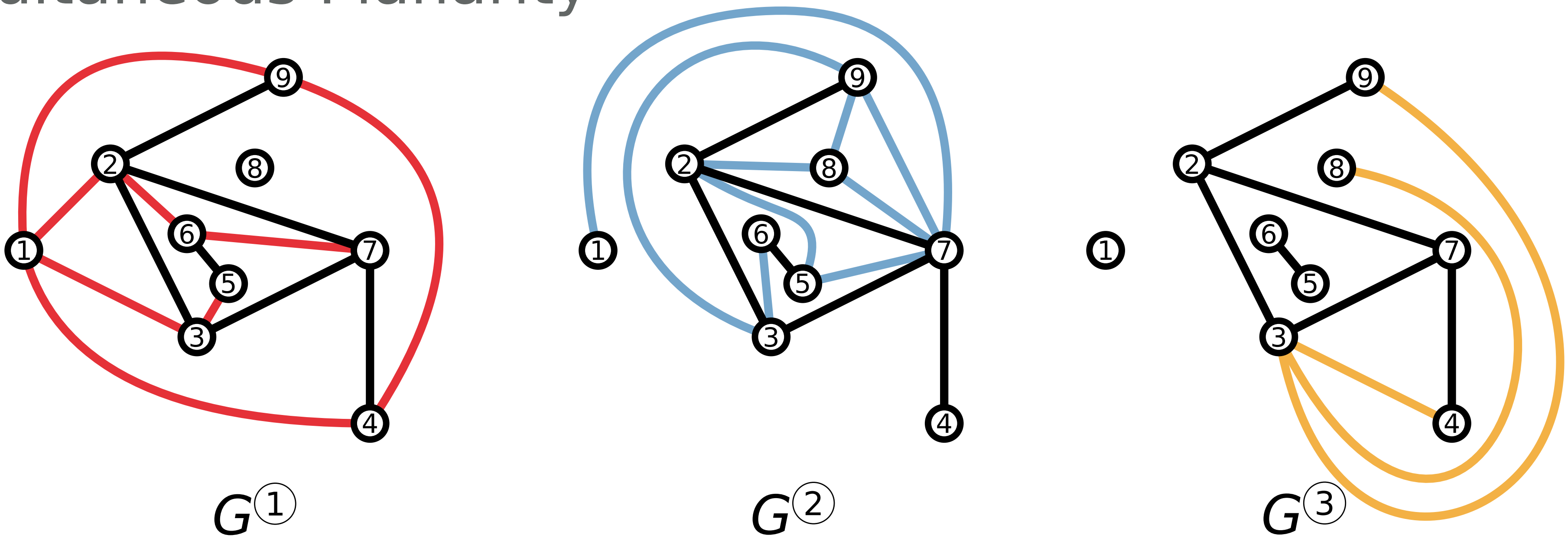

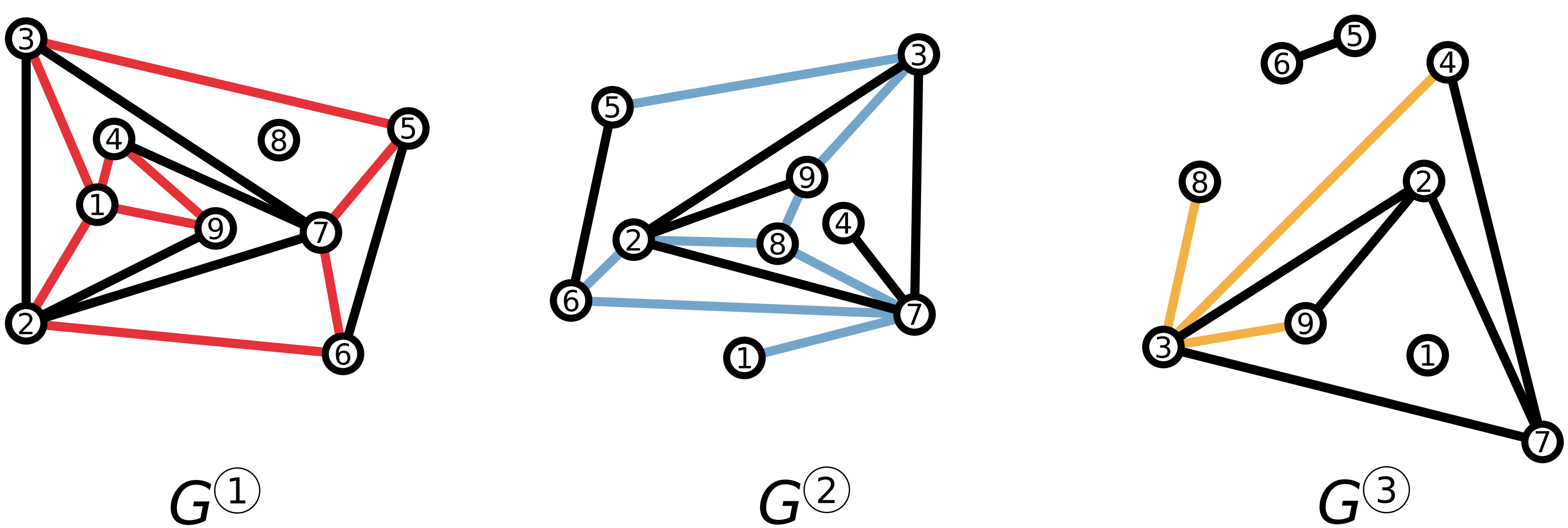

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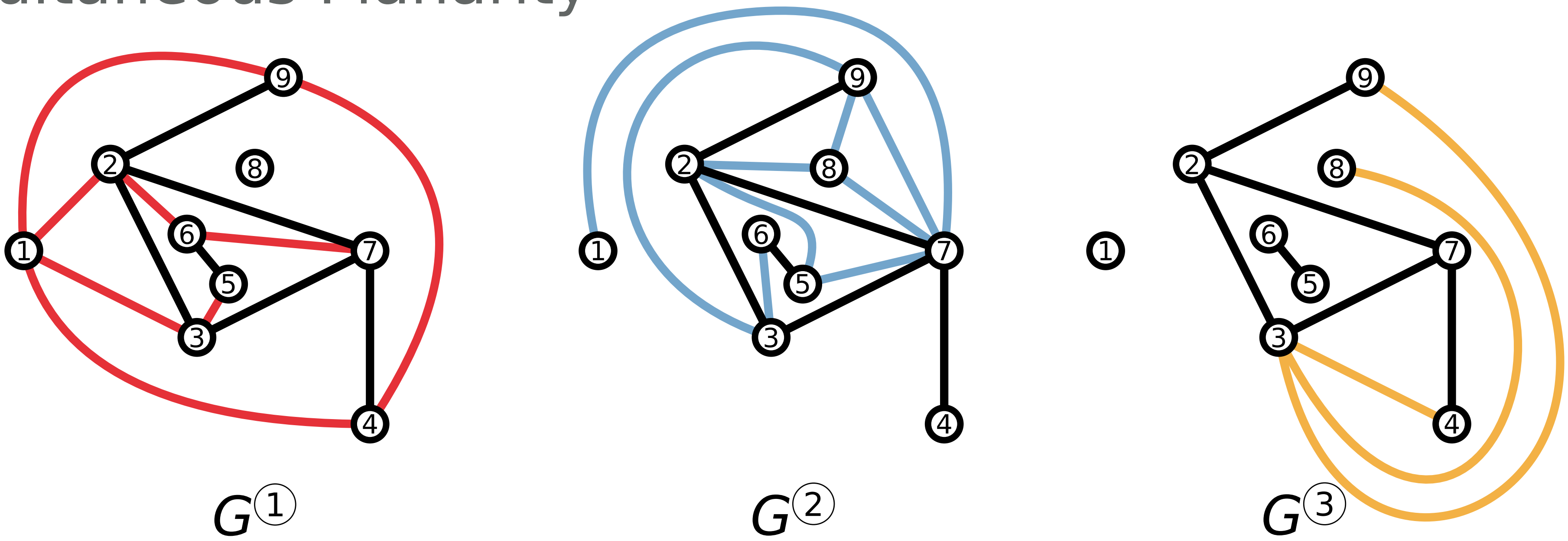
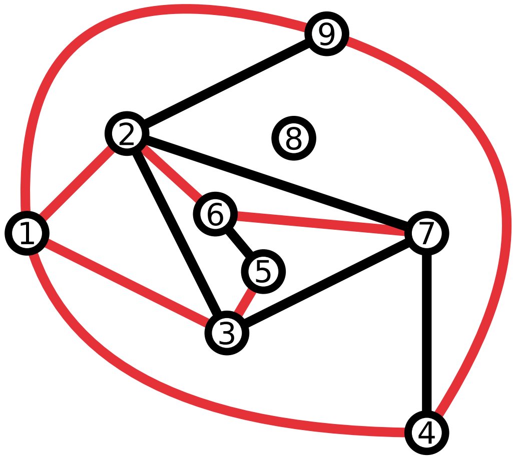


SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE)

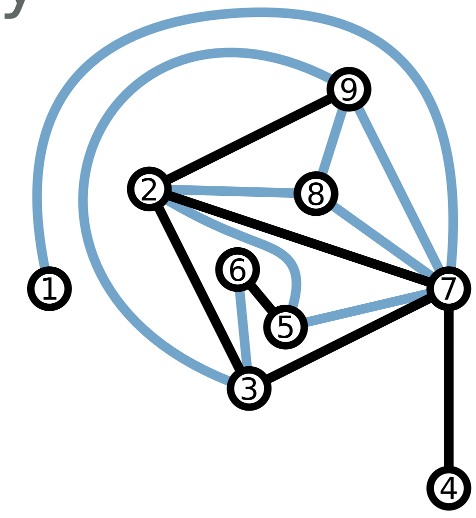
Input: k planar graphs $G^{①}, \dots, G^{ⓚ}$ on vertex set V with a *shared graph* $G^{ⓘ} \cap G^{ⓙ}$ for each pair.

Question: Are there planar drawings $\Gamma^{①}, \dots, \Gamma^{ⓚ}$ such that $\Gamma^{ⓘ}$ and $\Gamma^{ⓙ}$ coincide on $G^{ⓘ} \cap G^{ⓙ}$?

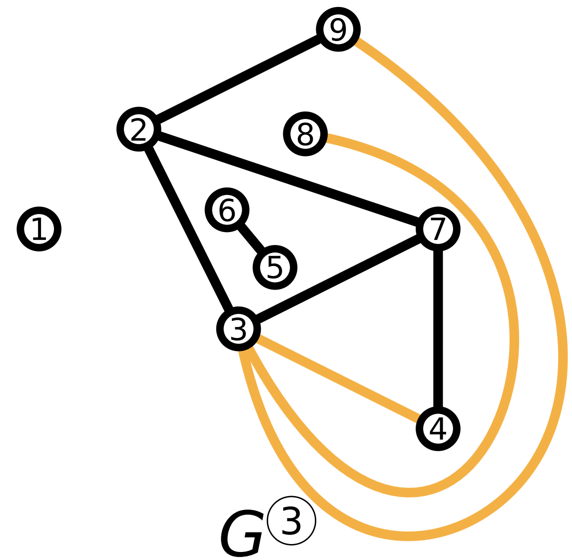
Simultaneous Planarity



$G^{(1)}$



$G^{(2)}$



$G^{(3)}$

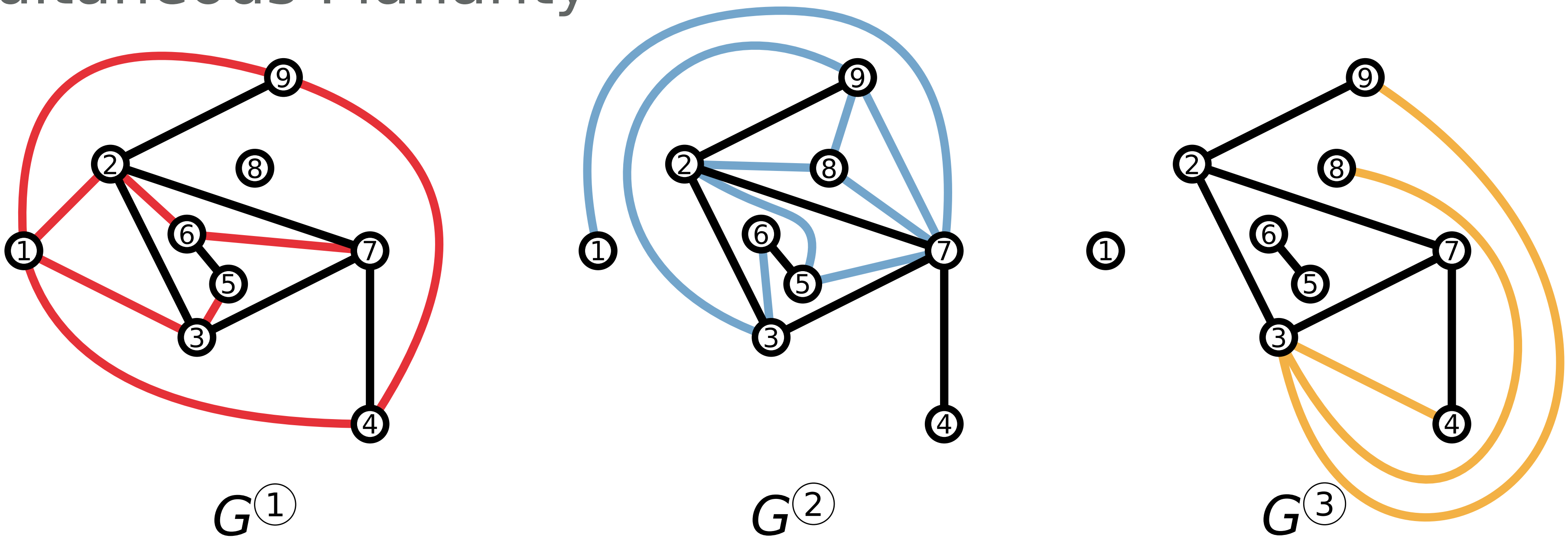
SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE)

Input: k planar graphs $G^{(1)}, \dots, G^{(k)}$ on vertex set V with a *shared graph* $G^{(i)} \cap G^{(j)}$ for each pair.

Question: Are there planar drawings $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ such that $\Gamma^{(i)}$ and $\Gamma^{(j)}$ coincide on $G^{(i)} \cap G^{(j)}$?

- **Sunflower Case:** $G^{(i)} \cap G^{(j)}$ is the same graph for each pair i, j

Simultaneous Planarity



SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE)

Input: k planar graphs $G^{\textcircled{1}}, \dots, G^{\textcircled{k}}$ on vertex set V with a *shared graph* $G^{\textcircled{i}} \cap G^{\textcircled{j}}$ for each pair.

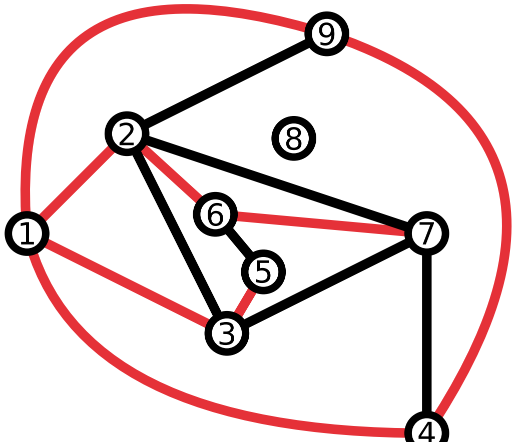
Question: Are there planar drawings $\Gamma^{\textcircled{1}}, \dots, \Gamma^{\textcircled{k}}$ such that $\Gamma^{\textcircled{i}}$ and $\Gamma^{\textcircled{j}}$ coincide on $G^{\textcircled{i}} \cap G^{\textcircled{j}}$?

- **Sunflower Case:** $G^{\textcircled{i}} \cap G^{\textcircled{j}}$ is the same graph for each pair i, j

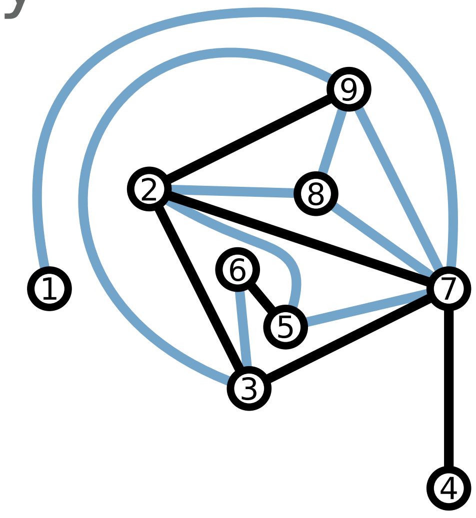
Theorem

SEFE is NP-complete for $k \geq 3$, [Gassner et al. '06]
even in the sunflower case. [Schaefer '13]

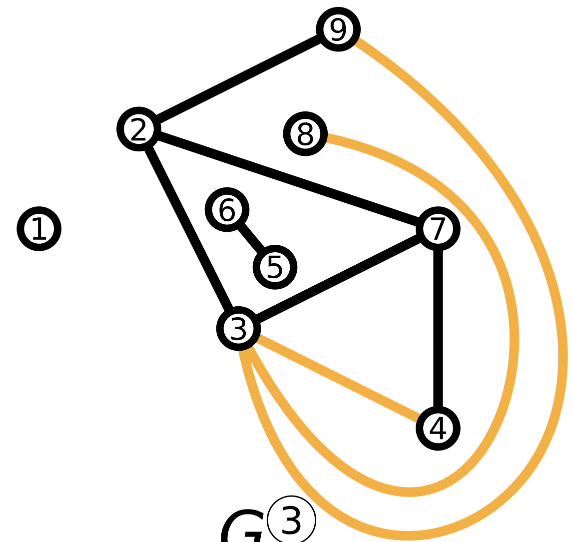
Simultaneous Planarity



$G^{①}$

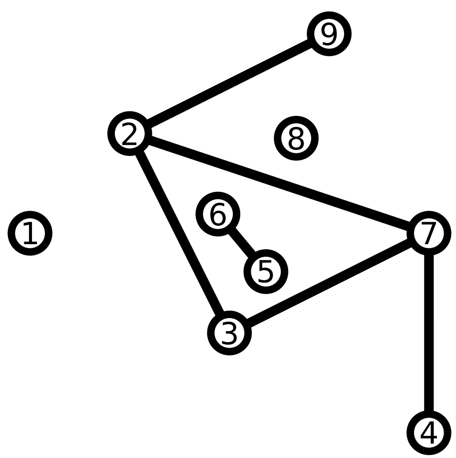


$G^{②}$



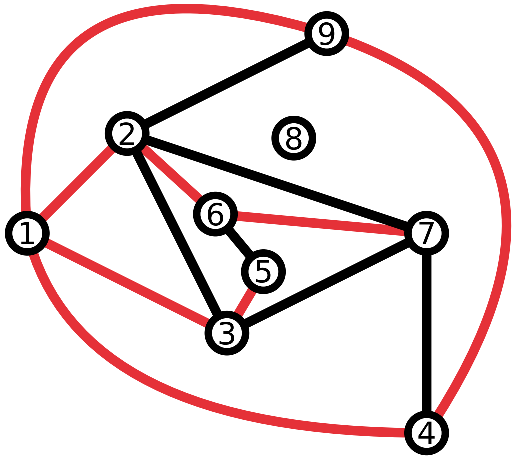
$G^{③}$

shared graph

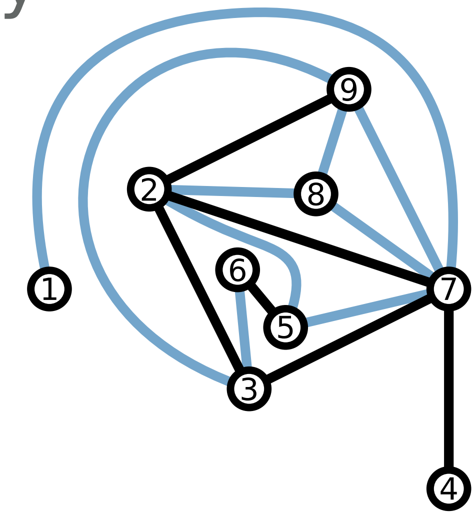


G

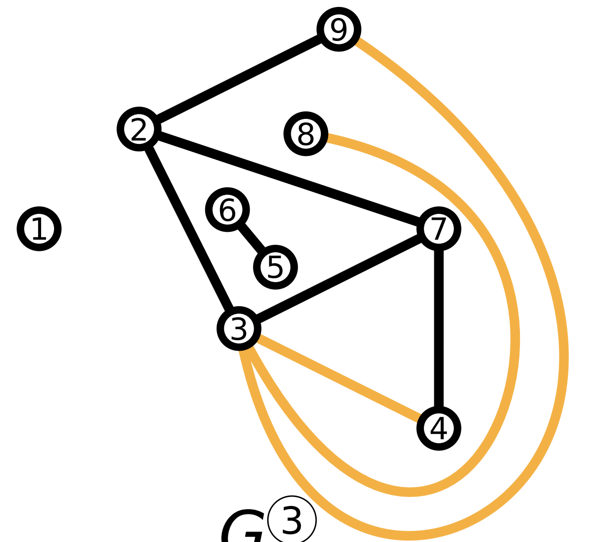
Simultaneous Planarity



$G^{\textcircled{1}}$

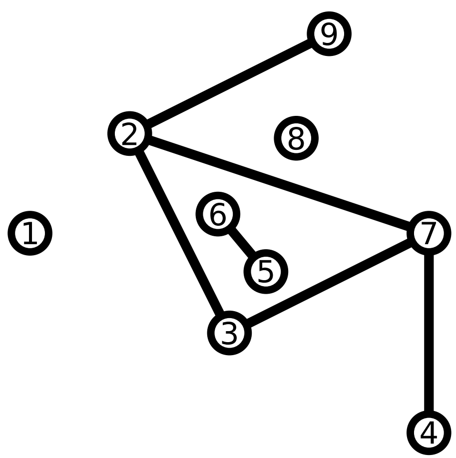


$G^{\textcircled{2}}$



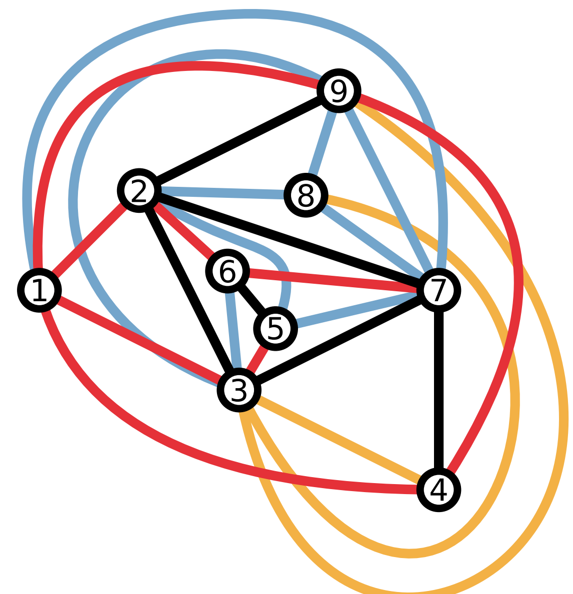
$G^{\textcircled{3}}$

shared graph



G

union graph

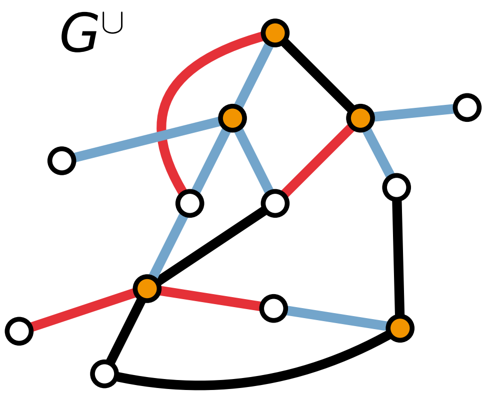


$G^{\cup}$

FPT-Algorithm: Vertex Cover Number of $G^{\cup}$

Vertex Cover C:
Every edge is incident to a vertex of C

Vertex Cover Number φ:
Size of a minimum Vertex Cover



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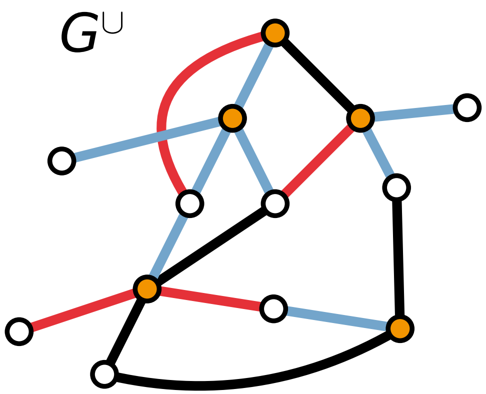
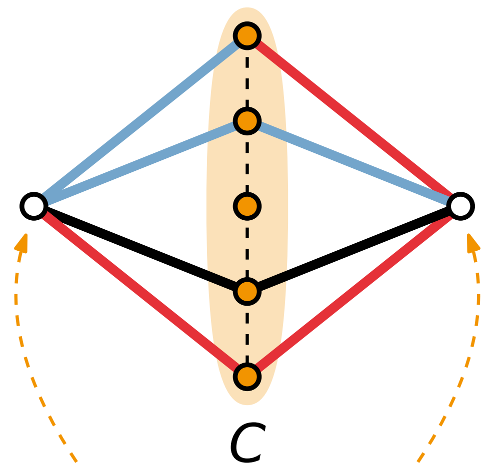
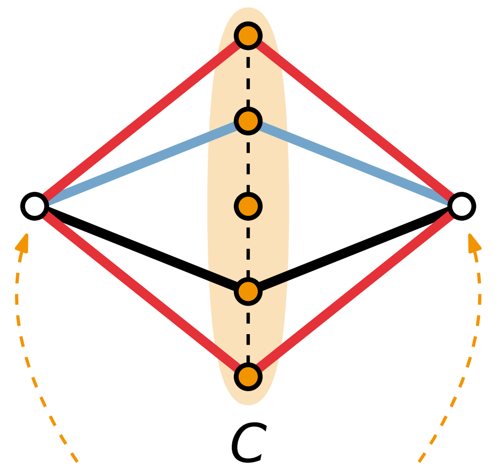


Partition vertices in $V(G^{\cup}) \setminus C$ into **types** according to the edges connecting them to C



different type



same type

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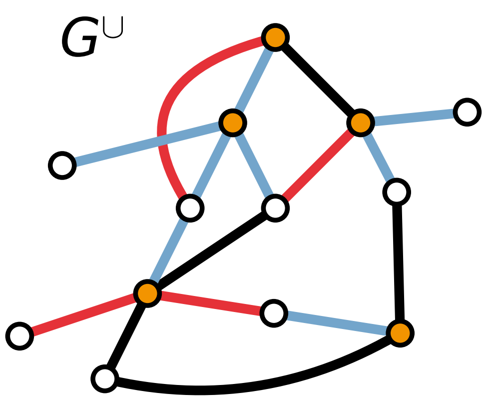

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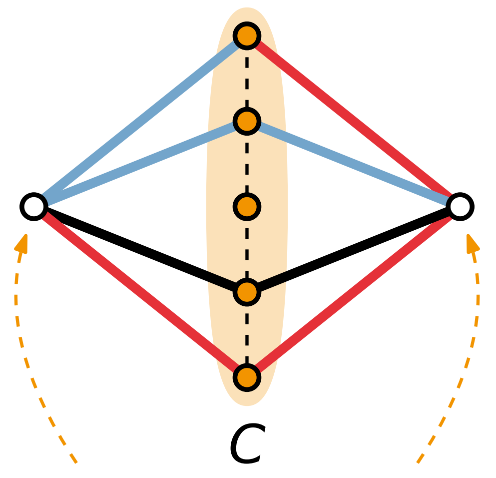

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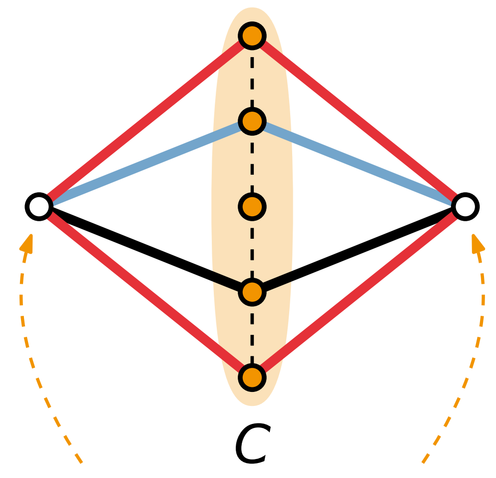
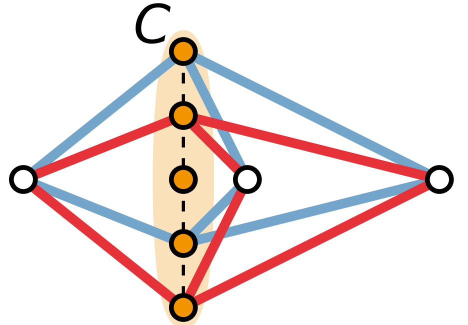


same type

$\mathcal{P}_{\leq 2}$:=
types with degree ≤ 2 in all $G^{(i)}$



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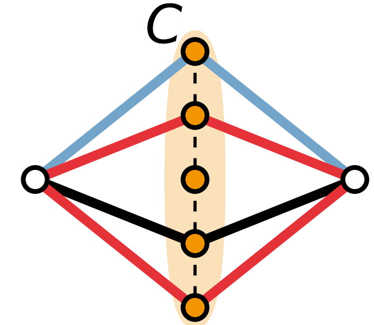

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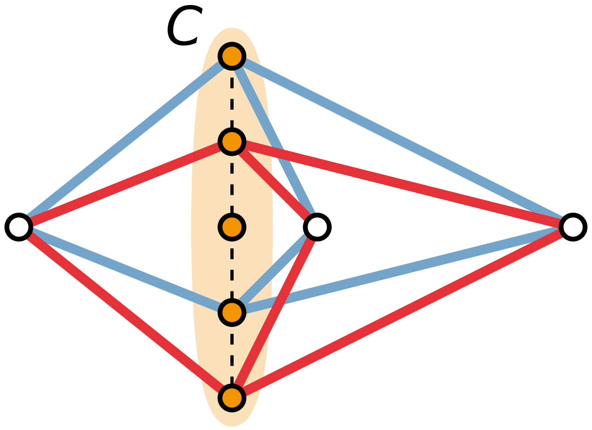
Reduction Rule 1: Type $U \in \mathcal{P}_{\leq 2}$ with $|U| > 1 \rightarrow$ remove one vertex of U



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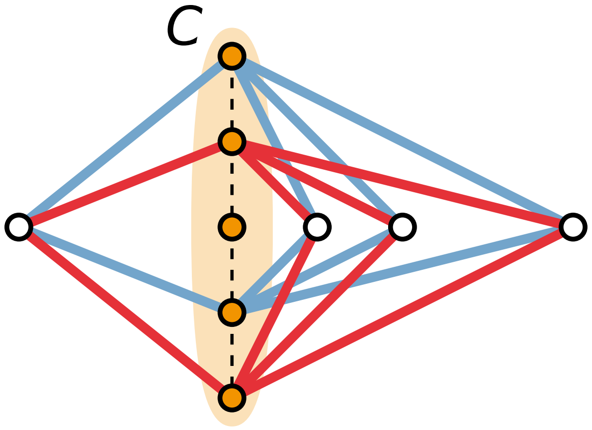

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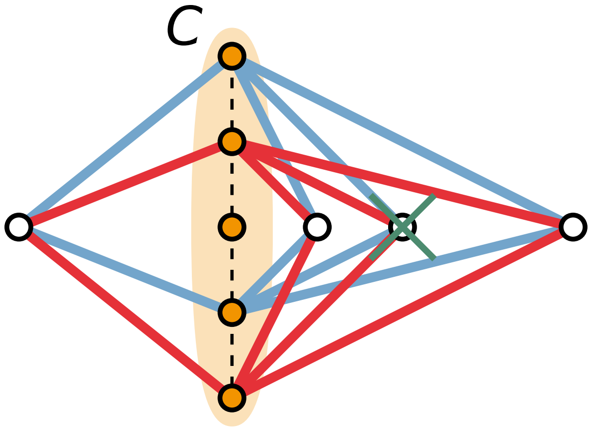

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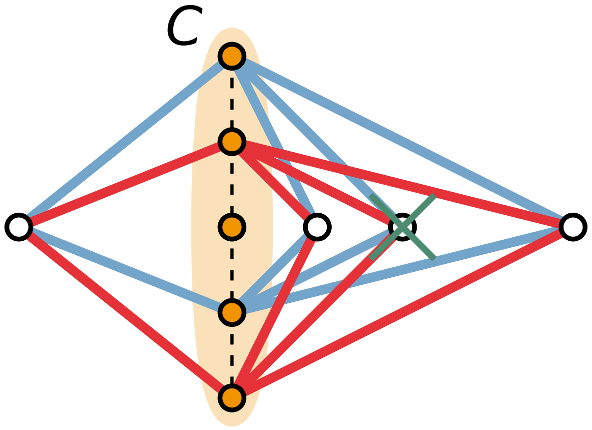

- **RR1:** Each type contains ≤ 1 vertices

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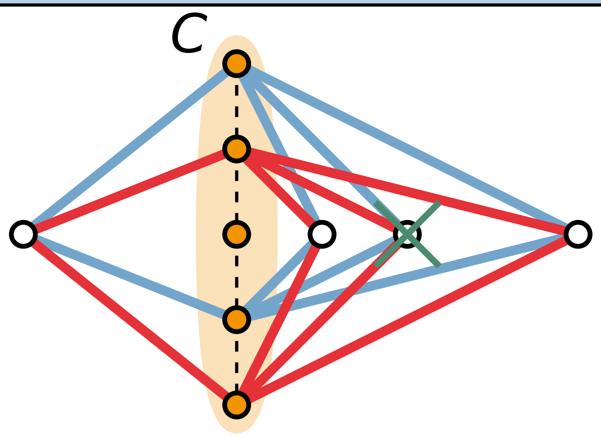

- **RR1:** Each type contains ≤ 1 vertices

  $\Rightarrow |\bigcup \mathcal{P}_{\leq 2}| \in O(\varphi^{2k})$

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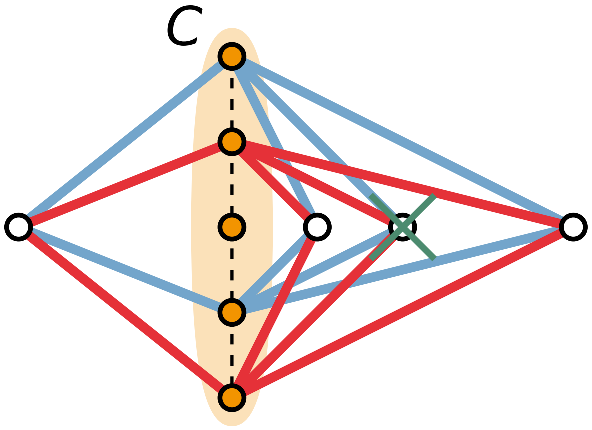

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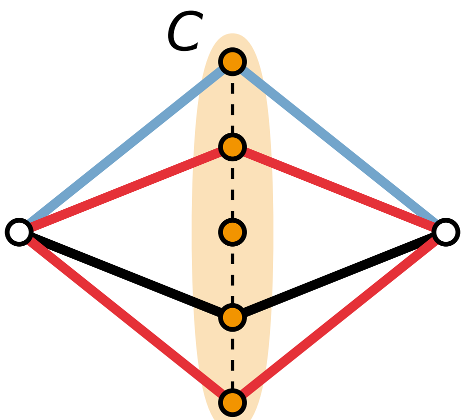
Remaining: vertices with degree ≥ 3 in some $G^{ⓘ}$



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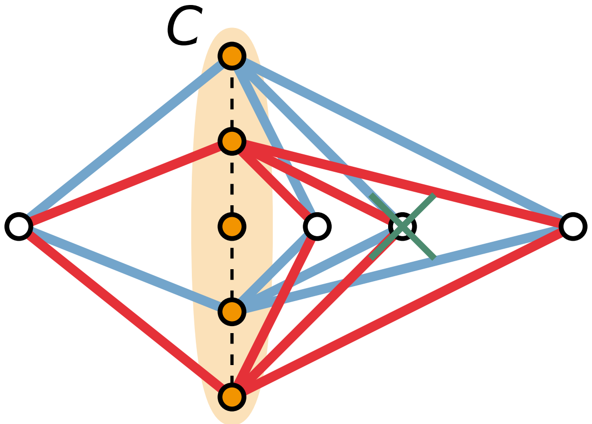

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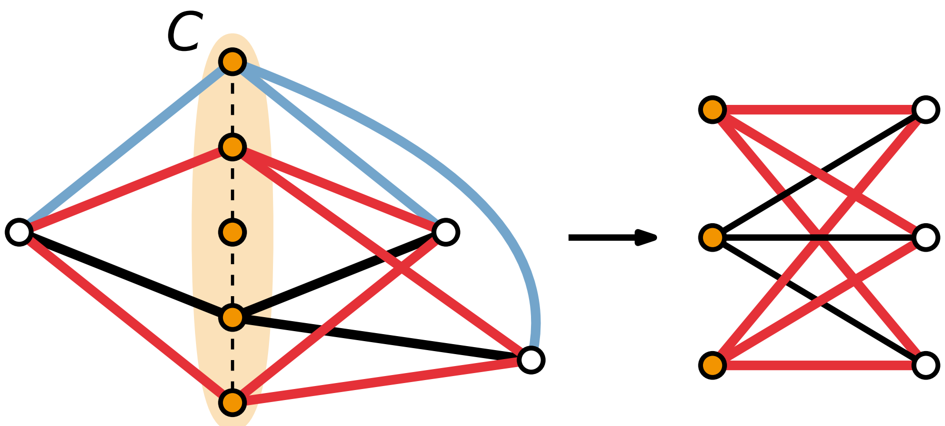

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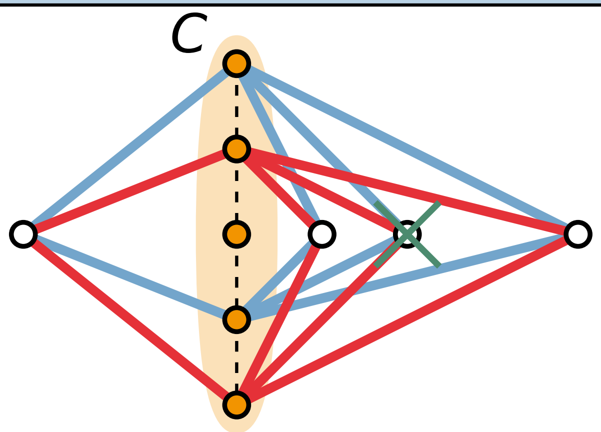

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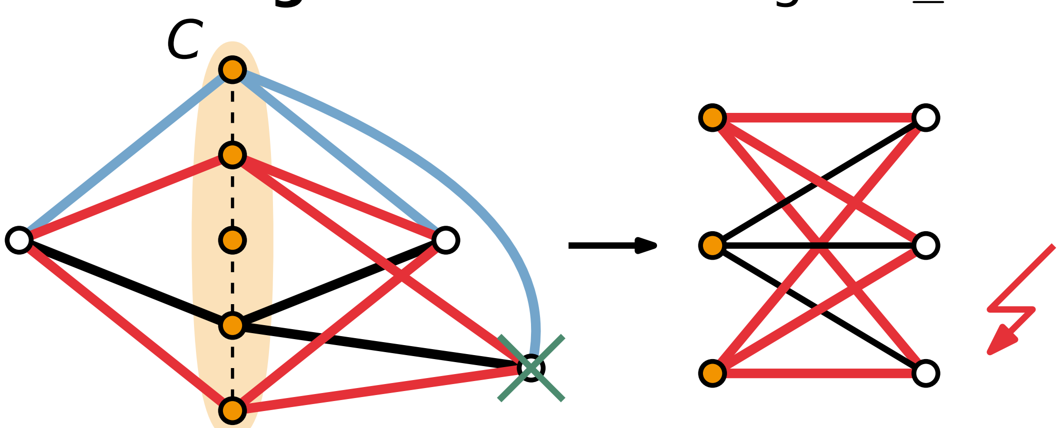

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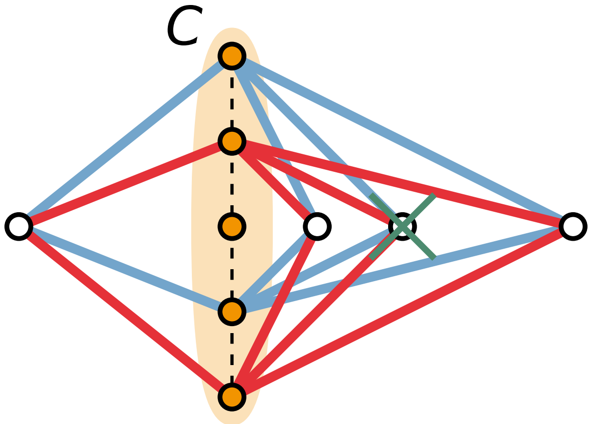

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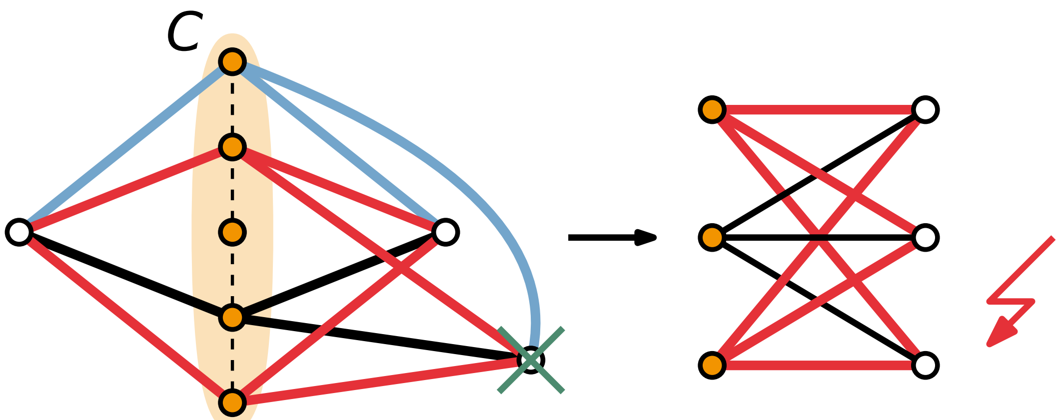

Planarity ensures:
#deg-$_{\geq}$3 vertices in $G^{(i)} \in O(\varphi)$

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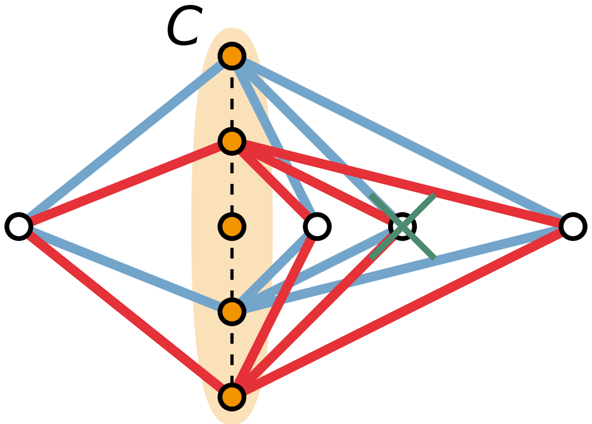

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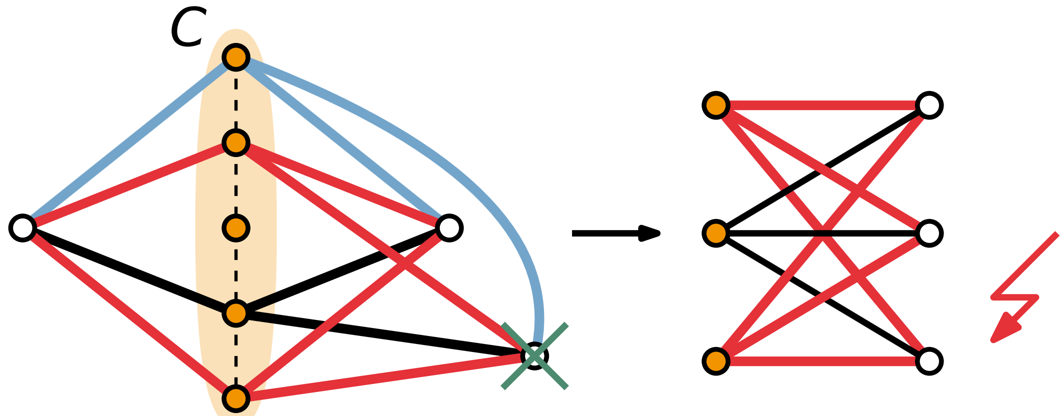
Remaining: vertices with degree ≥ 3 in some $G^{(i)}$



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$\Rightarrow |\bigcup \mathcal{P}_{\geq 3}| = O(k\varphi)$

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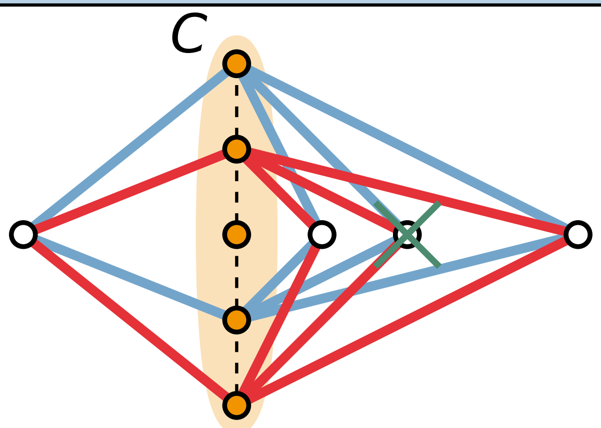

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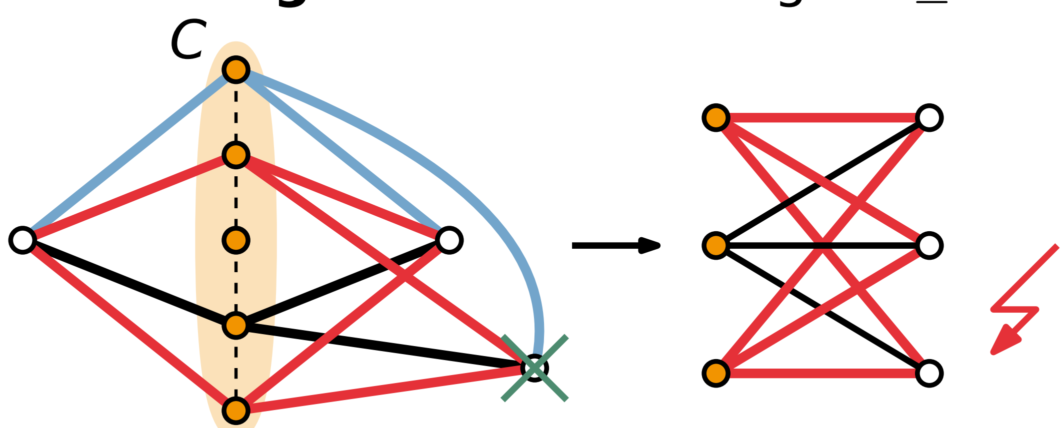

Planarity ensures:
#deg-$_{\geq}$3 vertices in $G^{ⓘ} \in O(\varphi)$

$\Rightarrow |\bigcup \mathcal{P}_{\geq 3}| = O(k\varphi)$

$\Rightarrow |V(G^{\cup})| = O(\varphi^{2k})$

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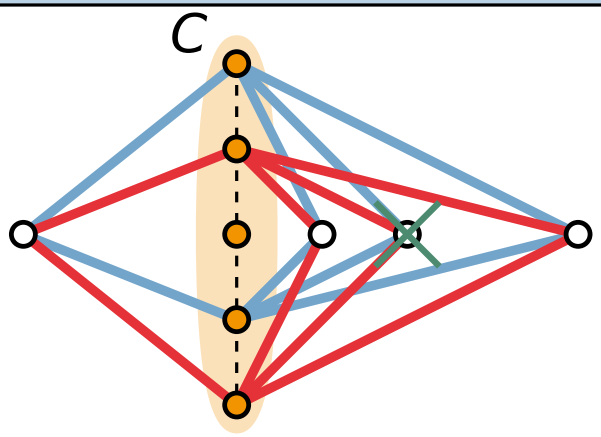

- **RR1:** Each type contains ≤ 1 vertices

  $\Rightarrow |\bigcup \mathcal{P}_{\leq 2}| \in O(\varphi^{2k})$

Remaining: vertices with degree ≥ 3 in some $G^{(i)}$

Theorem
SEFE is FPT w. r. t. the vertex cover number φ of $G^{\cup}$ and the number of input graphs k and admits a kernel of size $O(\varphi^{2k})$

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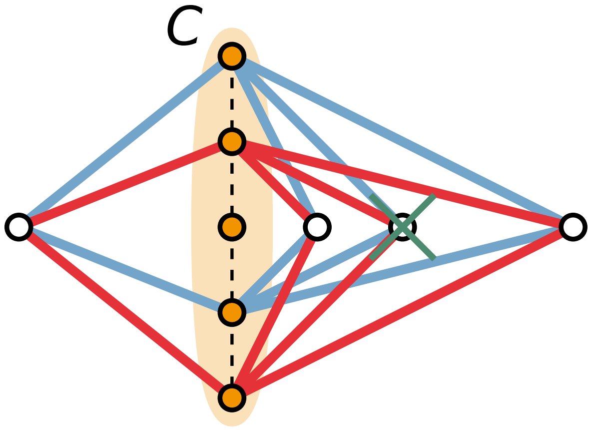


- **RR1:** Each type contains ≤ 1 vertices

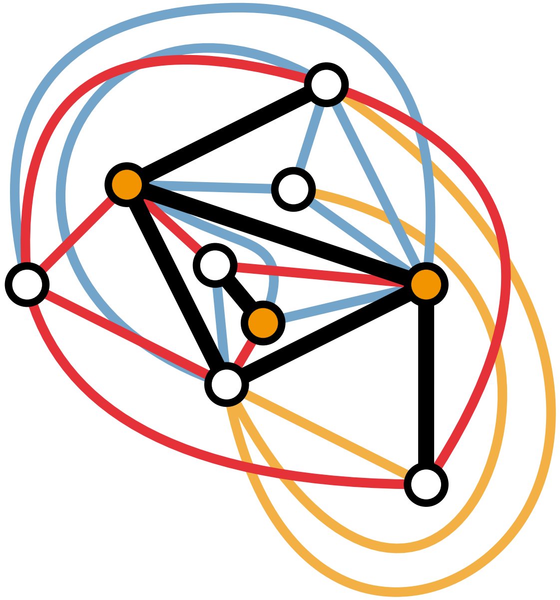
  $\Rightarrow |\bigcup \mathcal{P}_{\leq 2}| \in O(\varphi^{2k})$

Remaining: vertices with degree ≥ 3 in some $G^{ⓘ}$

Theorem
SEFE is FPT w. r. t. the vertex cover number φ of $G^{\cup}$ and the number of input graphs k and admits a kernel of size $O(\varphi^{2k})$

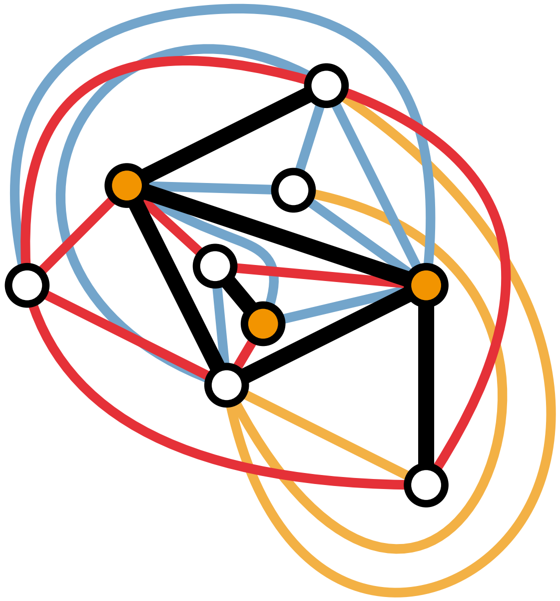
Theorem
SEFE is FPT w. r. t. the feedback edge set number ψ of $G^{\cup}$ and the number of input graphs k and admits a kernel of size $O(k\psi)$

Vertex Cover Number of *G*



What about the vertex cover number of the shared graph?

Vertex Cover Number of G



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Theorem [Angelini et al. '15]
SEFE is NP-complete for $k \geq 3$ input graphs, even if the shared graph is a star

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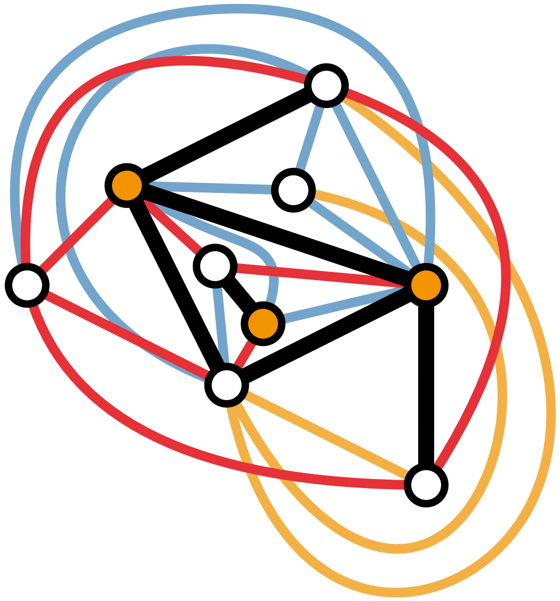

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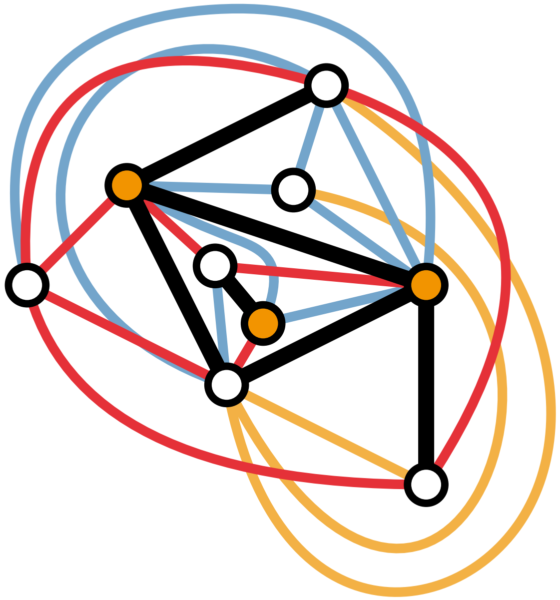
Vertex Cover Number of *G*



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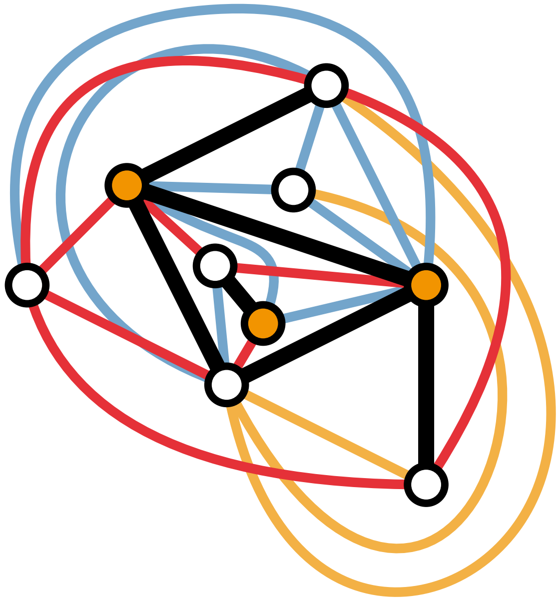
SEFE is FPT w.r.t. the vertex cover number plus the maximum degree of *G*

Vertex Cover Number of G



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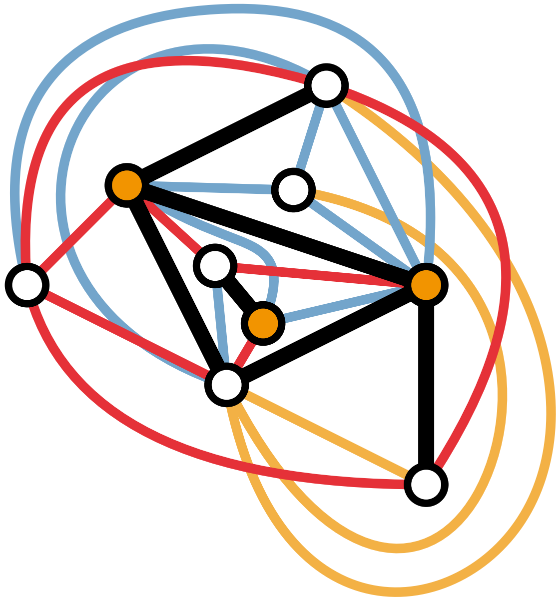
Theorem
SEFE is FPT w.r.t. the vertex cover number plus the maximum degree of G

Proof:
- combination bounds size of G (except for isolated vertices)
- brute-force all embeddings of G
- SEFE solvable in $O(n^2)$ if every conn. component of G has fixed embedding [Bläsius et al. ‘13]

Vertex Cover Number of G



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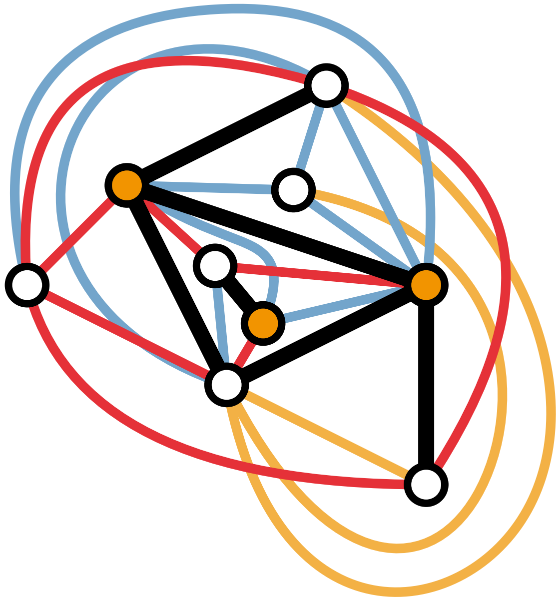
$\mathrm{vc}(G)$:= vertex cover number of G

Δ_1 := max. number of deg-1 neighbors of a single vertex in G

Vertex Cover Number of G



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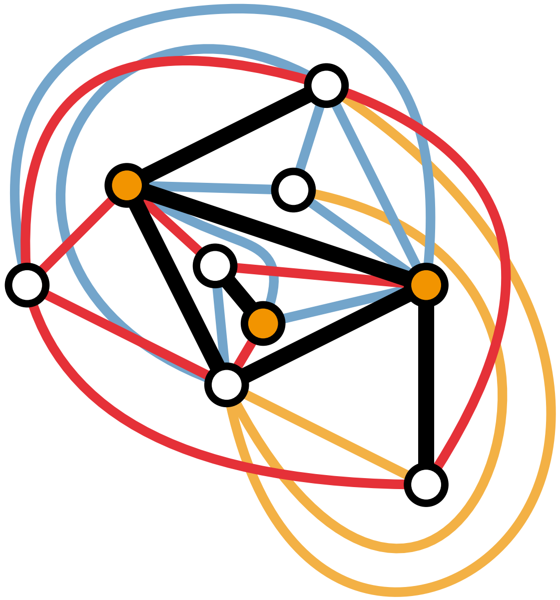

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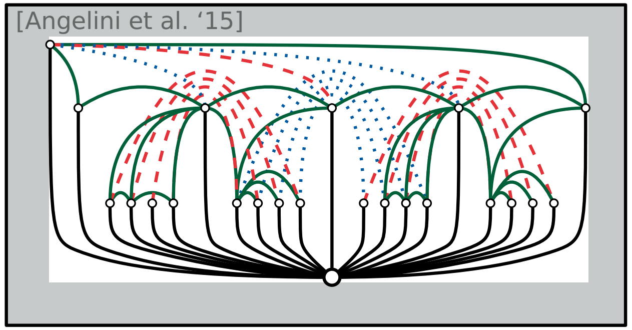
Theorem
SEFE is FPT w.r.t. $\mathrm{vc}(G) + \Delta_1$ and can be solved in time
$O(2^{O(\mathrm{vc}(G))} \cdot (2(\mathrm{vc}(G) + \Delta_1))^{(\mathrm{vc}(G)+\Delta_1)^2 \cdot 3\,\mathrm{vc}(G)} \cdot ((\mathrm{vc}(G) + \Delta_1)!)^{3\,\mathrm{vc}(G)} \cdot n^{O(1)})$

Parameters Shared Graph G



	vc(G)	fes(G)	cc(G)	cv(G)	Δ_1	Δ
vc(G)	✗					
fes(G)	✗	✗				
cc(G)	✗	✗	✗			
cv(G)	✗	✗	✗	✗		
Δ_1	✓	?	?	?	?	
Δ	✓	?	?	?	?	?

[this work]

Theorem
SEFE FPT w.r.t. vc(G) + Δ_1

vc(G) := vertex cover number
fes(G) := feedback edge set number
cc(G) := number of connected components
cv(G) := number of cutvertices
Δ_1 := max. number of deg-1 neighbors in G
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Betweenness $\leq$ SEFE [Angelini et al. '15]

- Input: Ground set $\mathcal{X}$, set $\mathcal{T}$ of triplets over $\mathcal{X}$
- Seek: Linear order of $\mathcal{X}$ where $(x, y, z) \in \mathcal{T} \Rightarrow y$ lies between x and z

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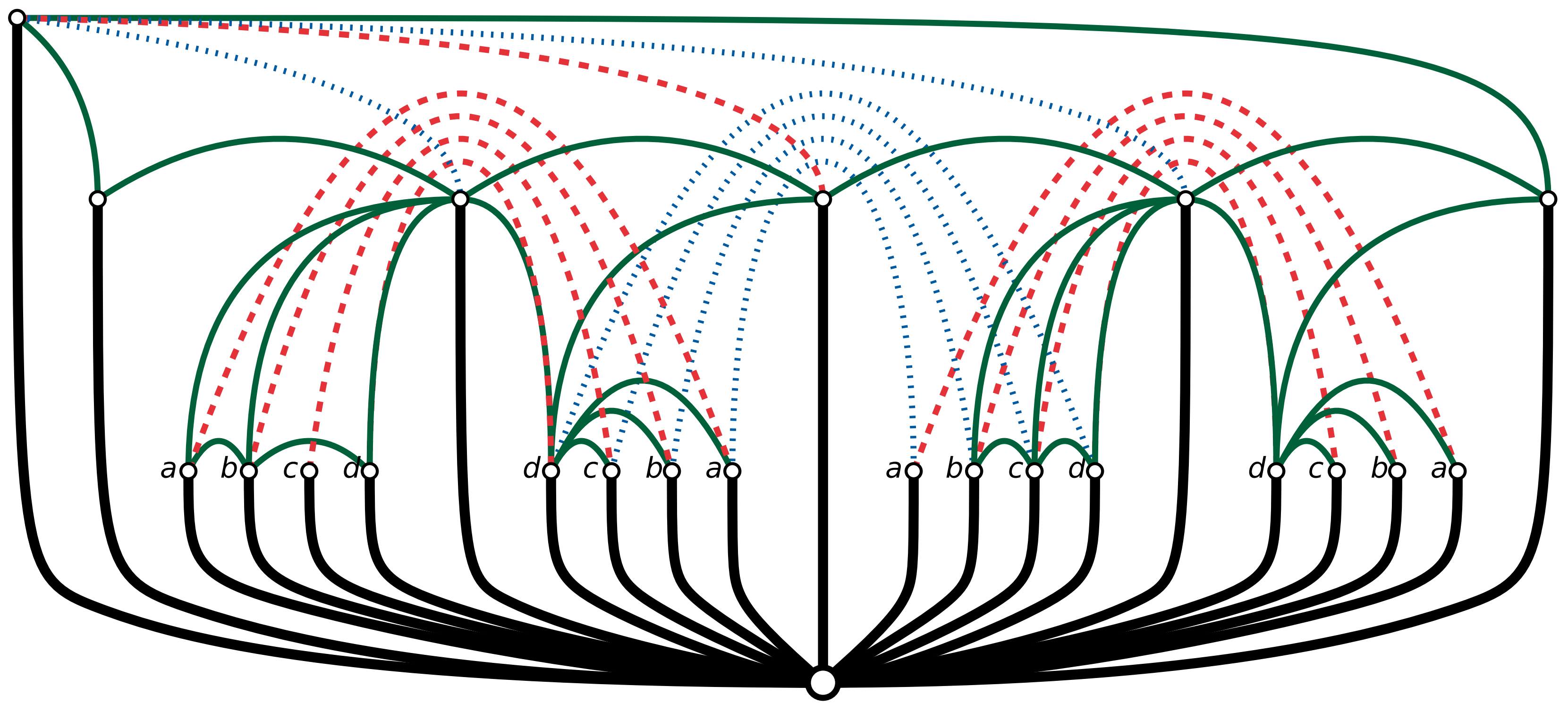

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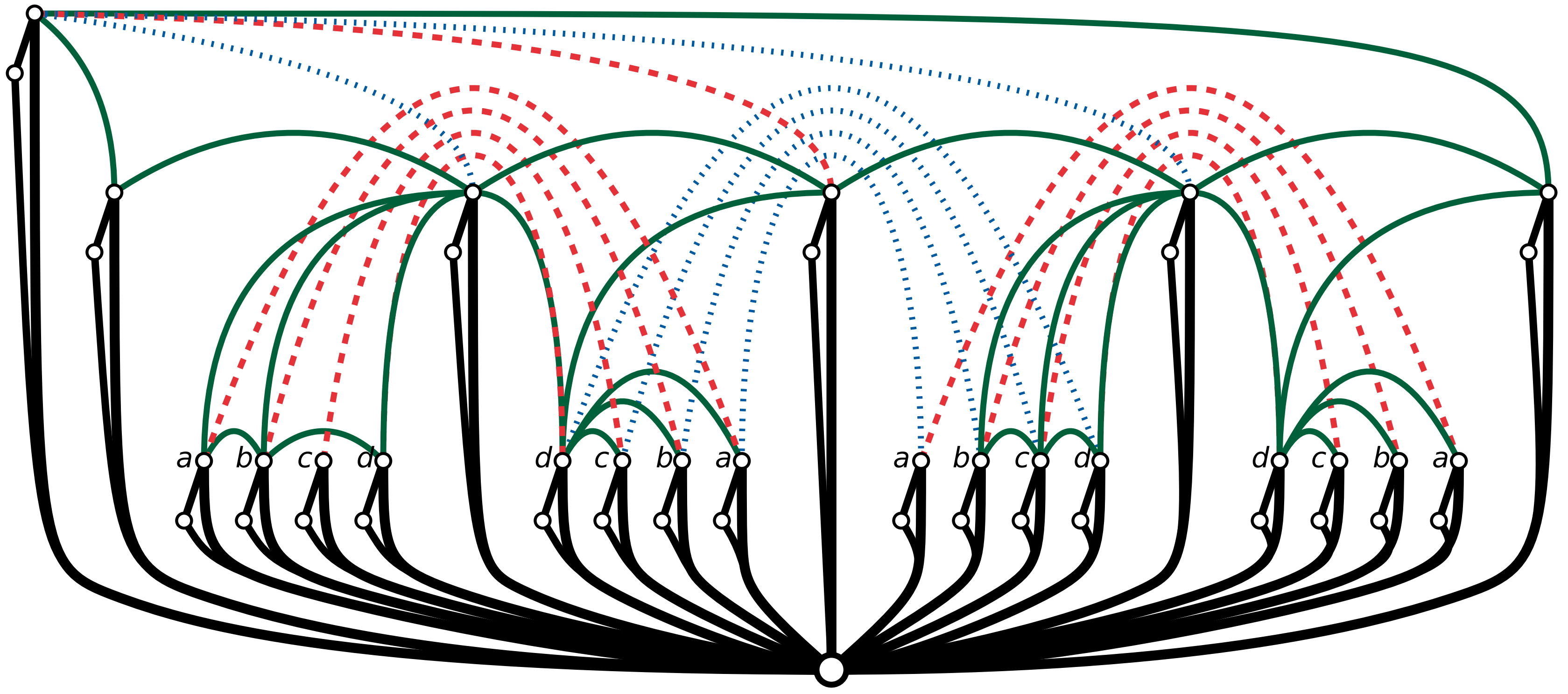

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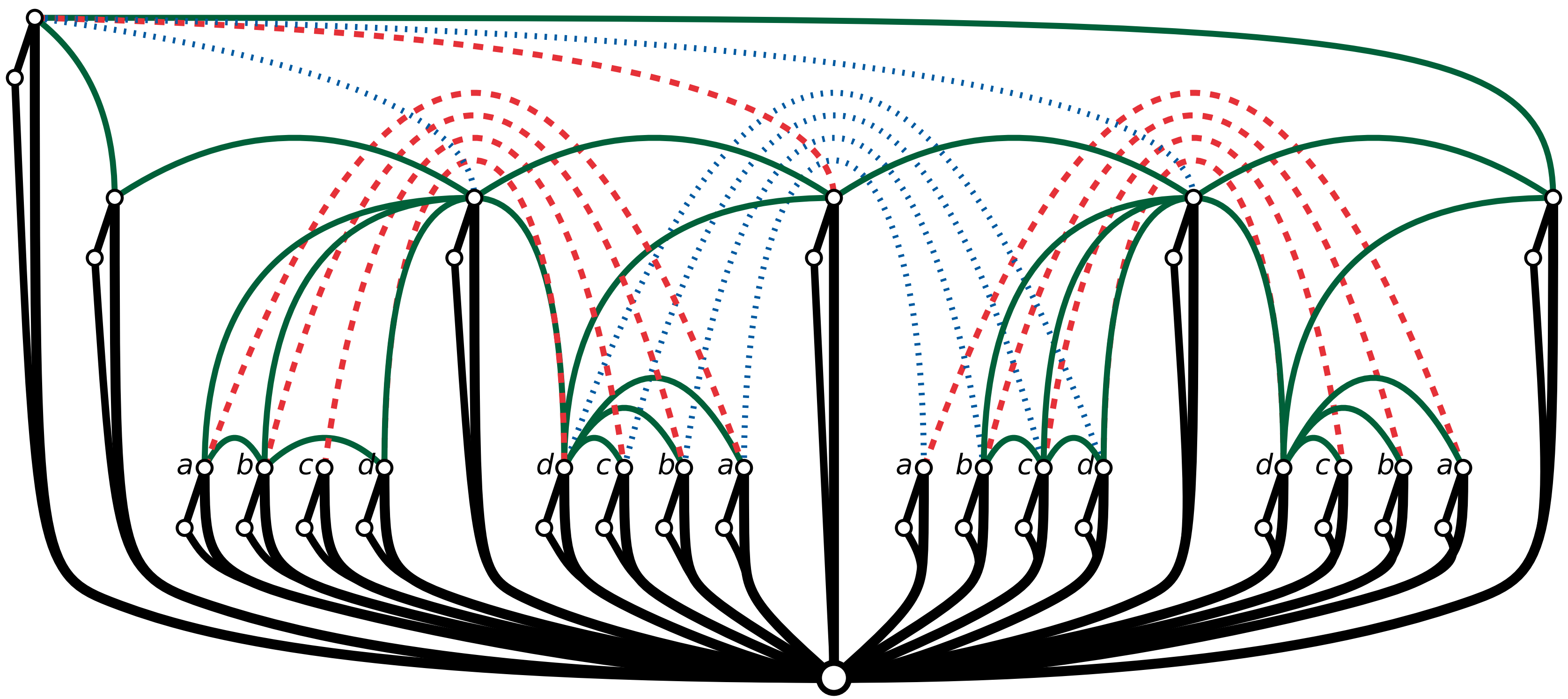
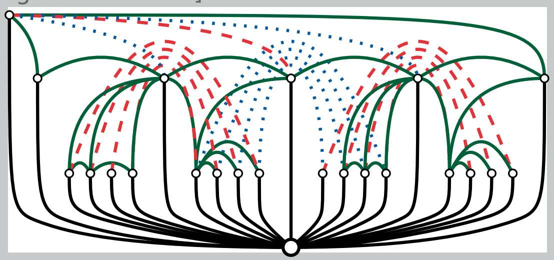


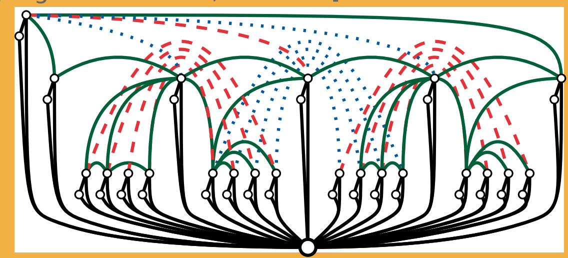
Δ_1 := max. number of deg-1 neighbors of a single vertex in G

Theorem
SEFE is para-NP-hard w.r.t. Δ_1

Parameters Shared Graph G



[Angelini et al. '15]



[Angelini et al. '15, modified]

	vc(G)	fes(G)	cc(G)	cv(G)	Δ_1	Δ
vc(G)	✗					
fes(G)	✗	✗				
cc(G)	✗	✗	✗			
cv(G)	✗	✗	✗	✗		
Δ_1	✓	?	✗	✗	✗	
Δ	✓	?	?	?	?	?

[this work]

Theorem
SEFE FPT w.r.t. vc(G) + Δ_1

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Reduction with small maximum degree

- Idea: Linear ordering of $\mathcal{X}$ via linear orderings of triplets

$$\mathcal{X} = \{a, b, c, d\}, \mathcal{T} = \{(a, b, c), (a, b, d)\}$$

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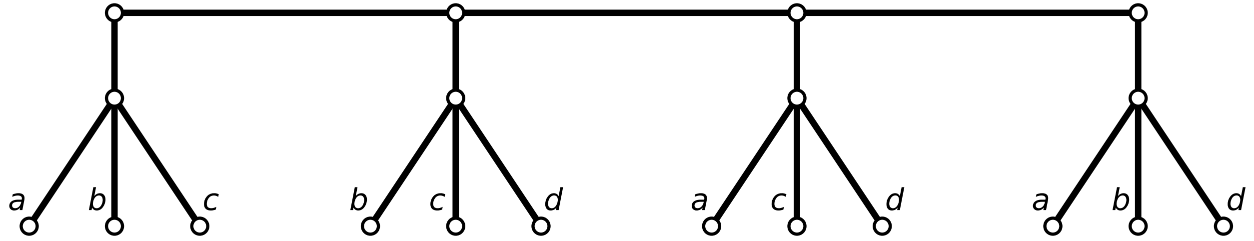

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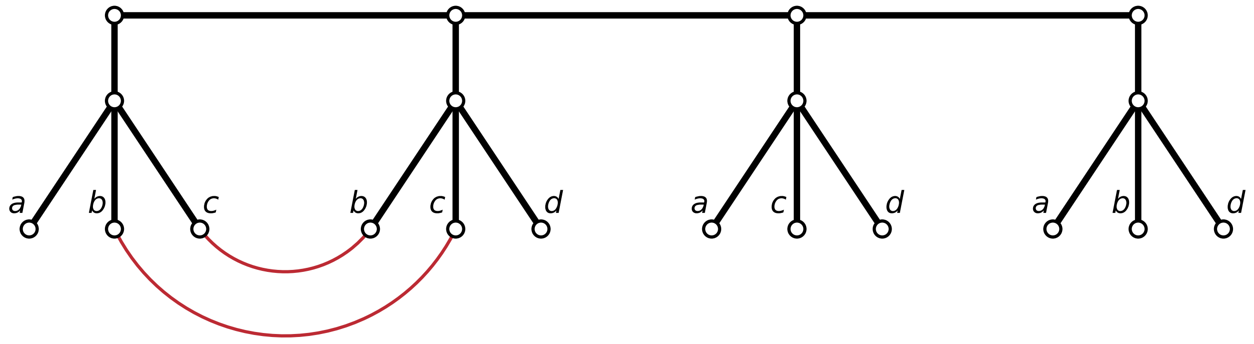
- Ensure consistency of pairs via exclusive edges
- Triplets enforce transitivity



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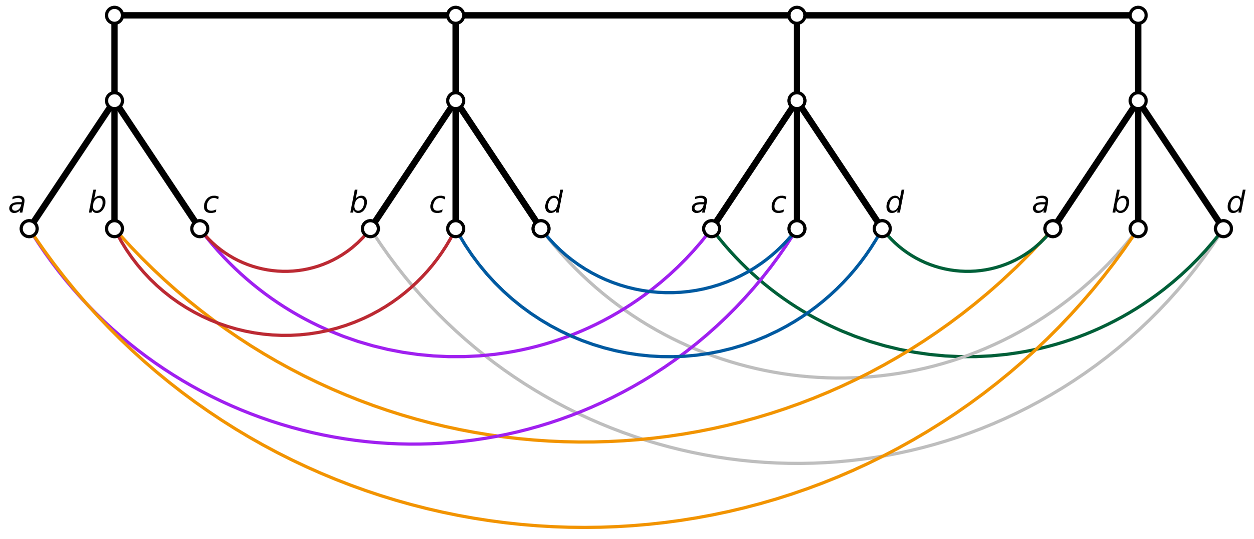

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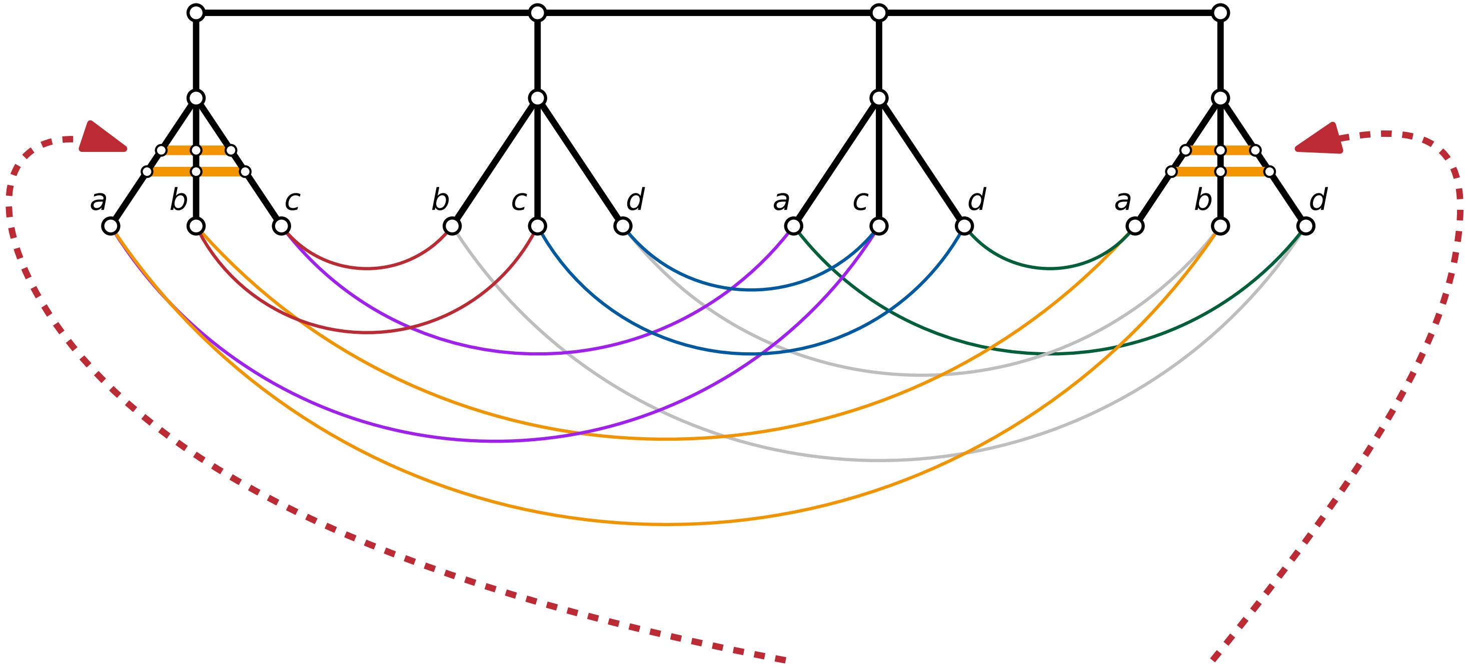

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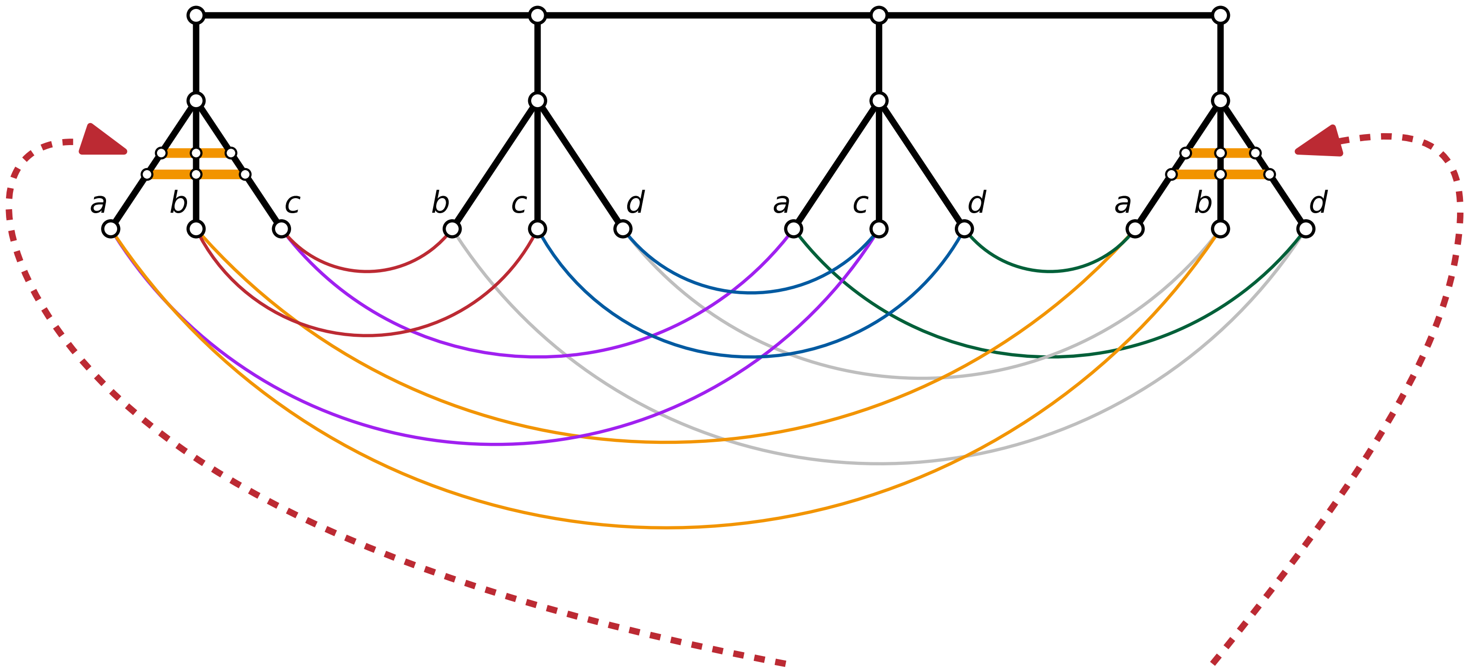

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Theorem
If the number of graphs is part of the input, SEFE is NP-complete, even if the shared graph is a tree with maximum degree 4

Reduction with small maximum degree

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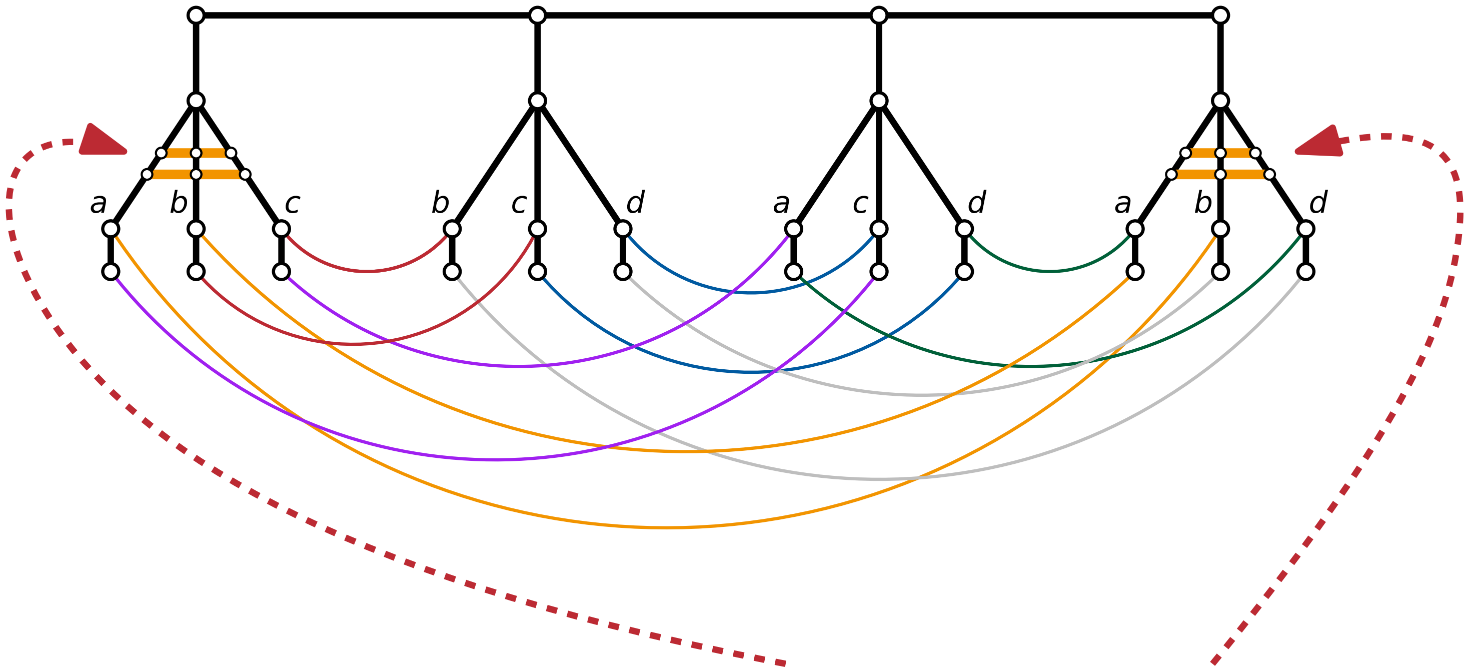

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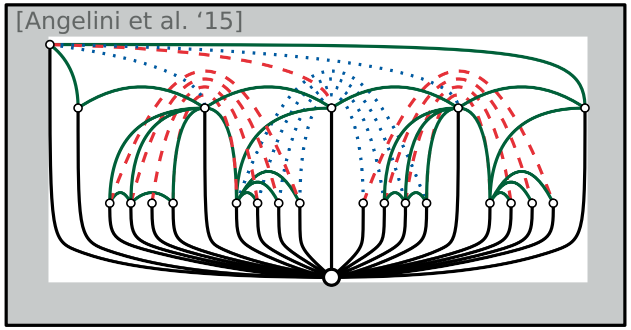
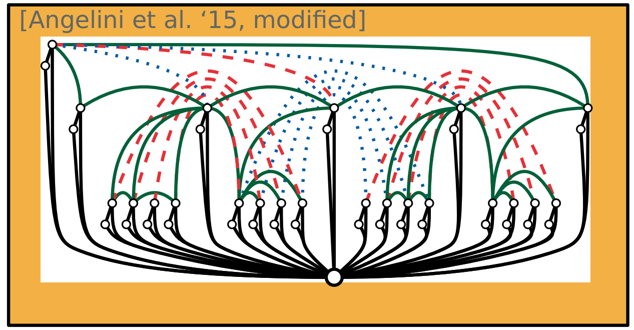
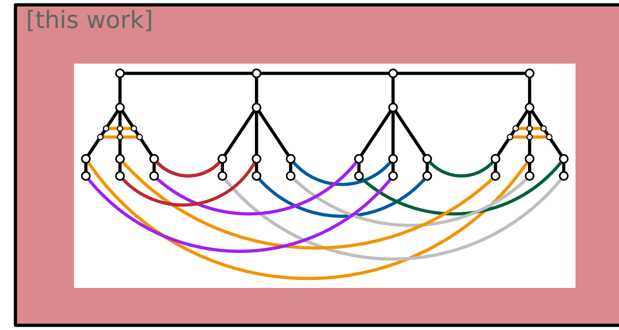
Theorem
If the number of graphs is part of the input, SEFE is NP-complete, even if the shared graph is a tree and the union graph has maximum degree 4

Overview Parameters Shared Graph*







	$\mathrm{vc}(G)$	$\mathrm{fes}(G)$	$\mathrm{cc}(G)$	$\mathrm{cv}(G)$	Δ_1	Δ	$\Delta^{(i)}$	$\Delta^{\cup}$
$\mathrm{vc}(G)$	✗							
$\mathrm{fes}(G)$	✗	✗						
$\mathrm{cc}(G)$	✗	✗	✗					
$\mathrm{cv}(G)$	✗	✗	✗	✗				
Δ_1	✓	✗	✗	✗	✗			
Δ	✓	✗	✗	✓	✗	✗		
$\Delta^{(i)}$	✓	✗	✗	✓	✗	✗	✗	
$\Delta^{\cup}$	✓	✗	✗	✓	✗	✗	✗	✗

[this work]

Theorem
SEFE FPT w.r.t. $\mathrm{vc}(G) + \Delta_1$

[this work]

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$\Delta^{(i)}$:= max. degree among all input graphs
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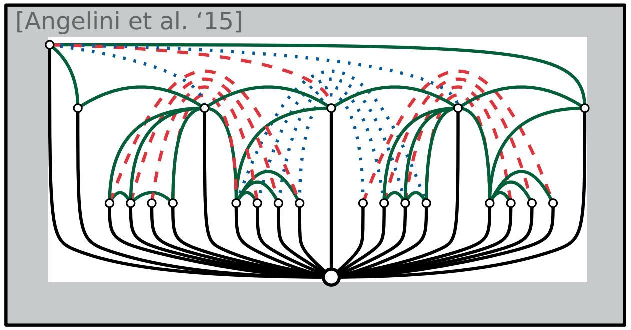

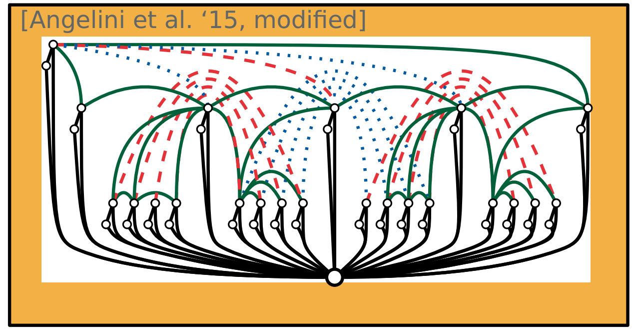

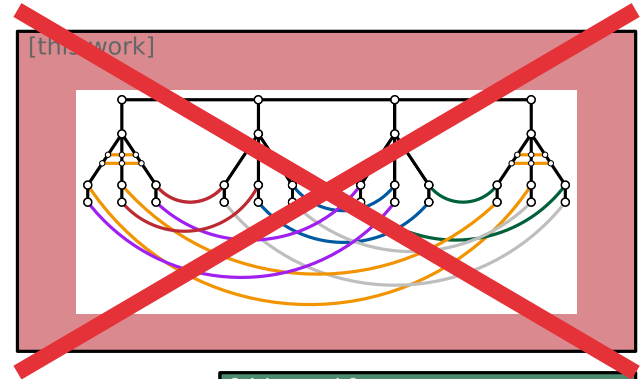

[this work]
Theorem
SEFE FPT w.r.t. vc(G) + Δ_1

[this work]
Theorem
SEFE FPT w.r.t. cv(G) + Δ

	vc(G)	fes(G)	cc(G)	cv(G)	Δ_1	Δ	$\Delta^{(i)}$	$\Delta^{\cup}$
vc(G)	✗							
fes(G)	✗	✗						
cc(G)	✗	✗	✗					
cv(G)	✗	✗	✗	✗				
Δ_1	✓	?	✗	✗	✗			
Δ	✓	?	?	✓	?	?		
$\Delta^{(i)}$	✓	?	?	✓	?	?	?	
$\Delta^{\cup}$	✓	?	?	✓	?	?	?	?

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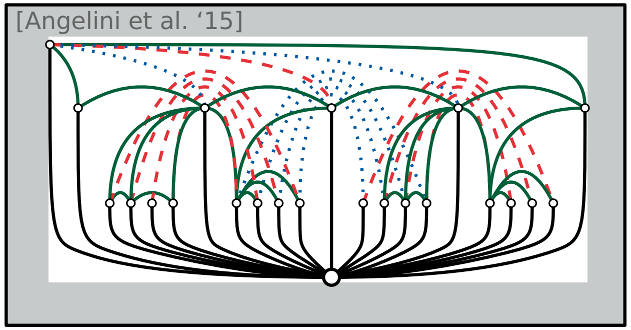

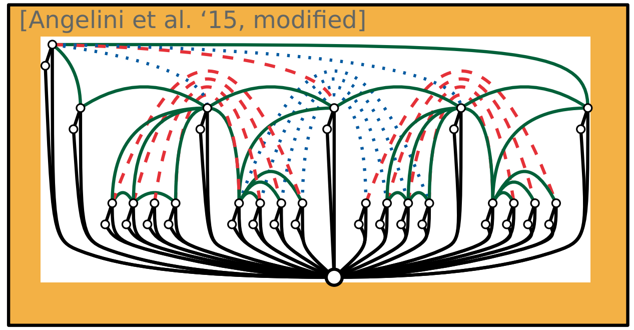

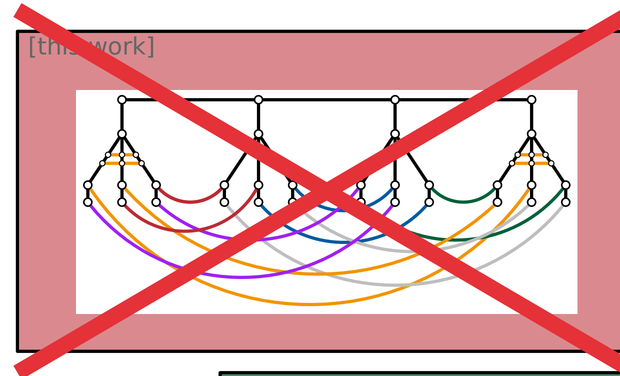

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[this work]

Theorem
SEFE FPT w.r.t. $\mathrm{cv}(G) + \Delta$

$\mathrm{vc}(G)$	✗							
$\mathrm{fes}(G)$	✗	✗						
$\mathrm{cc}(G)$	✗	✗	✗					
$\mathrm{cv}(G)$	✗	✗	✗	✗				
Δ_1	✓	?	✗	✗	✗			
Δ	✓	?	?	✓	?	?		
$\Delta^{(i)}$	✓	?	?	✓	?	?	?	
$\Delta^{\cup}$	✓	?	?	✓	?	?	?	?
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Is SEFE still hard for constant maximum degree of G and constant number of input graphs?

~~*If the number of graphs is part of the input~~