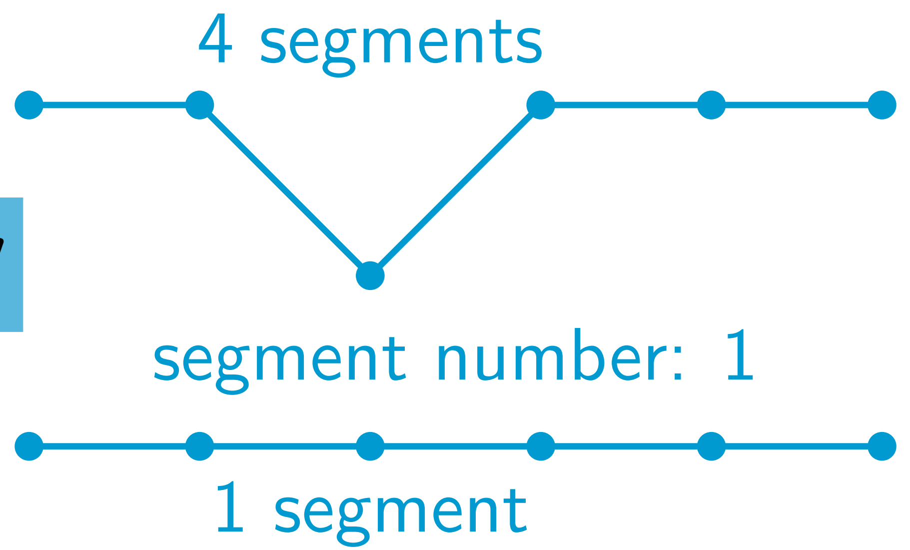
The Parametrized Complexity of the Segment Number



Sabine Cornelsen
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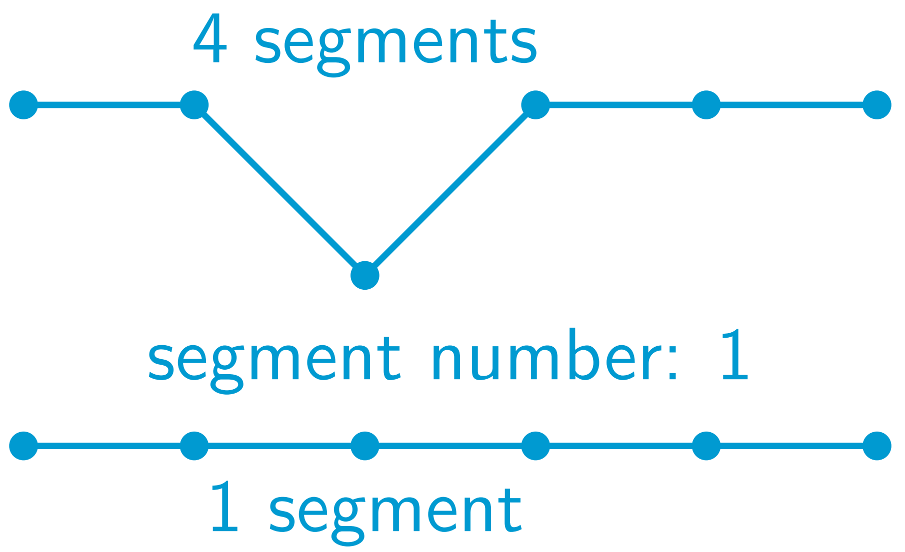
Würzburg, Germany

Symposium on Graph Drawing and Network Visualization, GD 2023, Isola delle Femmine

Basic Definitions

segment = maximal set of edges forming a line segment



segment number seg(G) of a planar graph G:
minimum number of segments
in any planar straight-line drawing of G

Dujmović, Eppstein, Suderman, Wood (CGTA'07)

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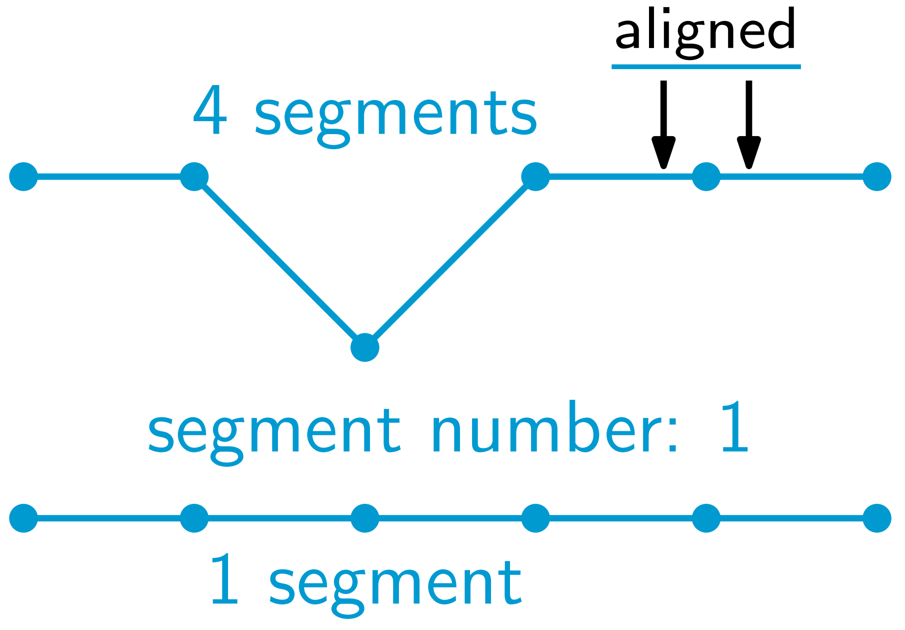

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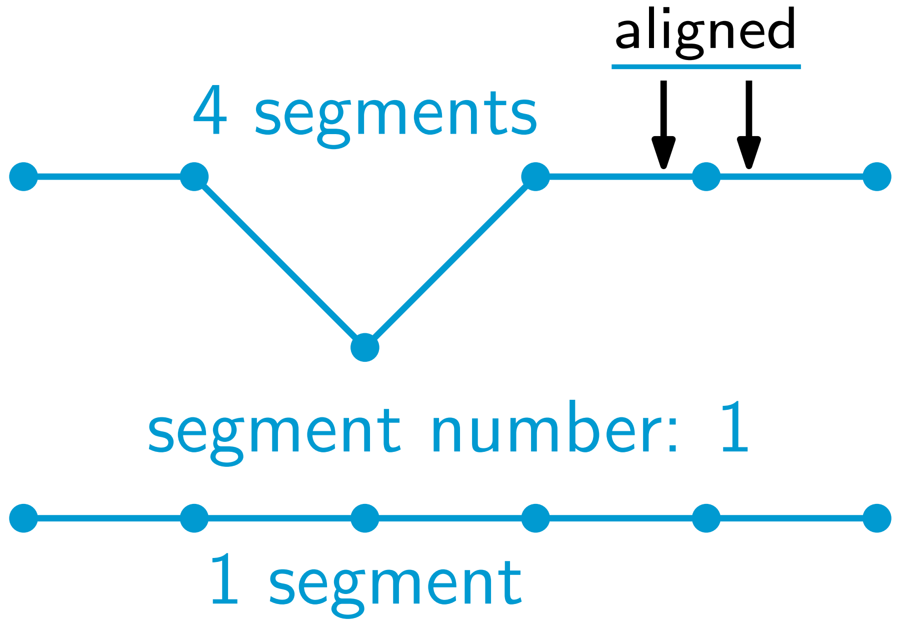

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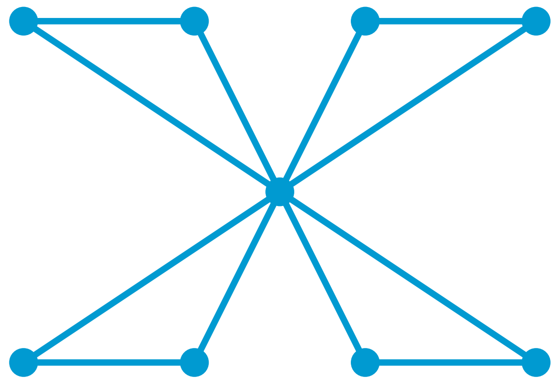

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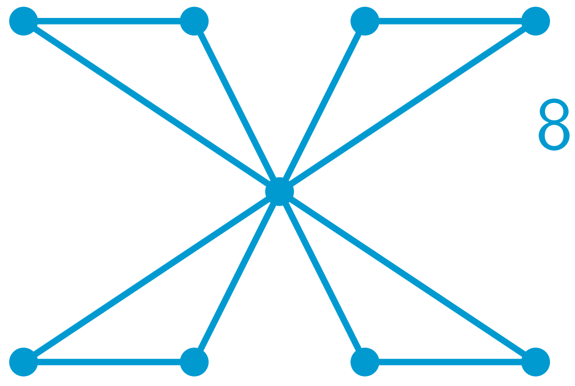

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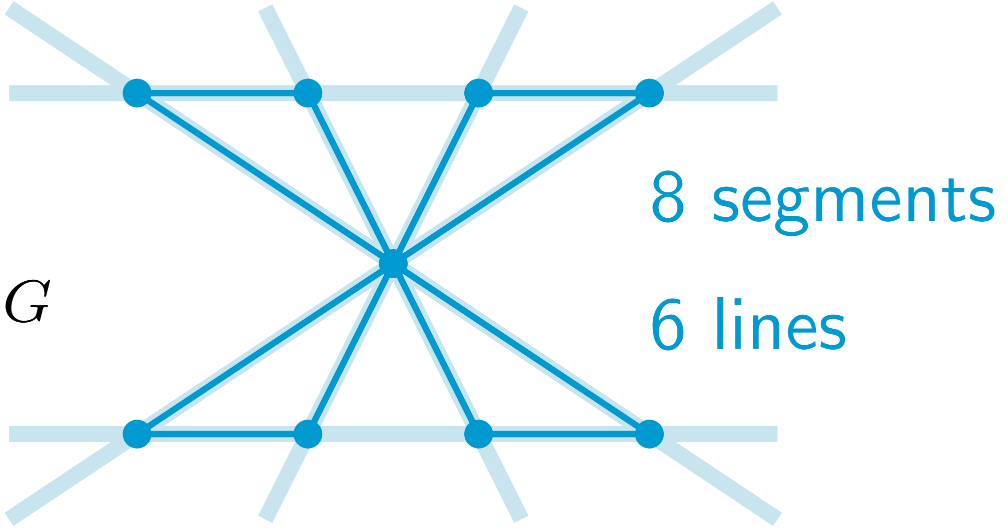
Chaplick et al. (GD'16)



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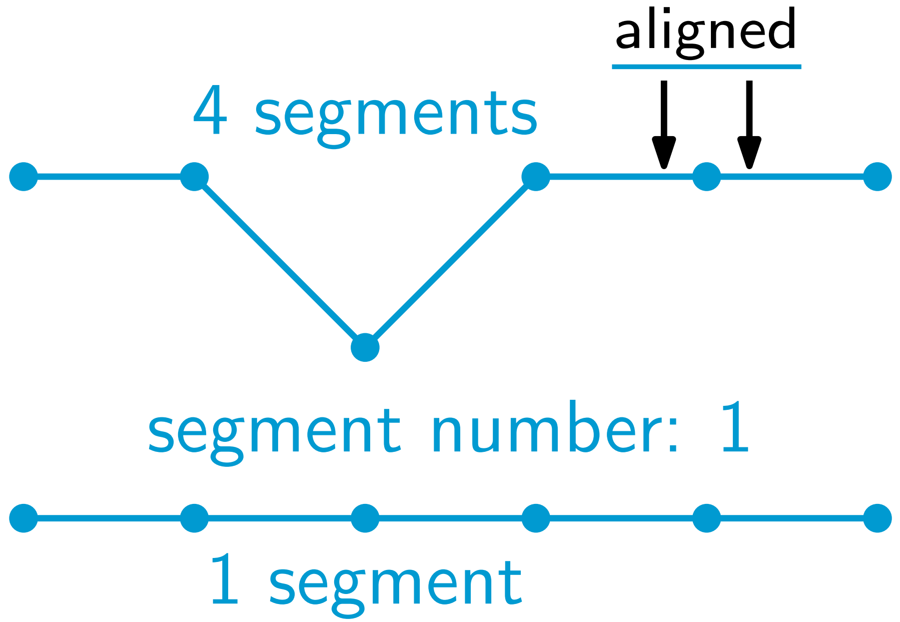

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respective optimization problem: Segment Number

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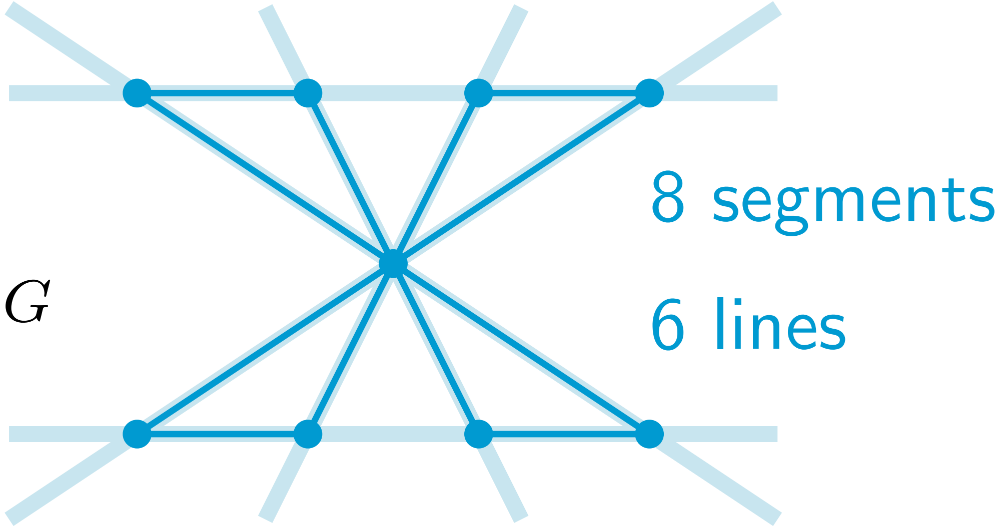
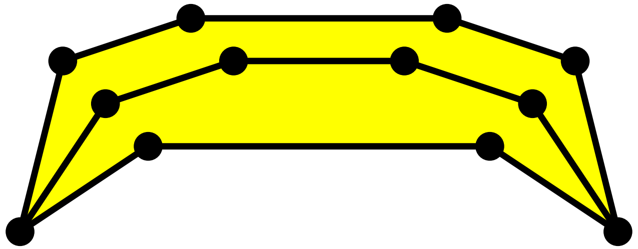


Warm-Up: Banana-Trees, and -Cycles



banana: Scott/Seymour 2020

union of internally disjoint paths with common endpoints

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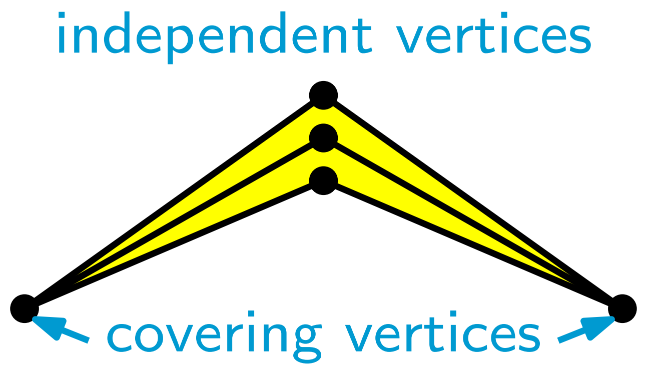

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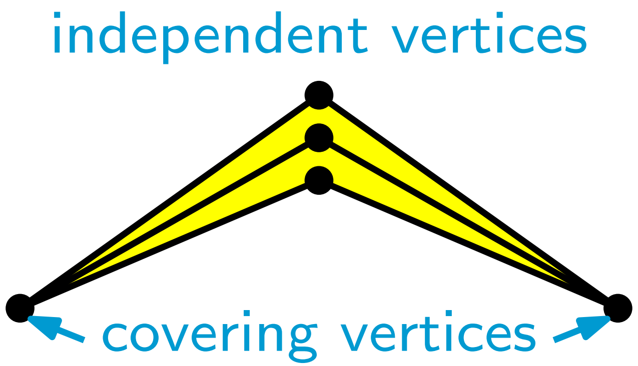

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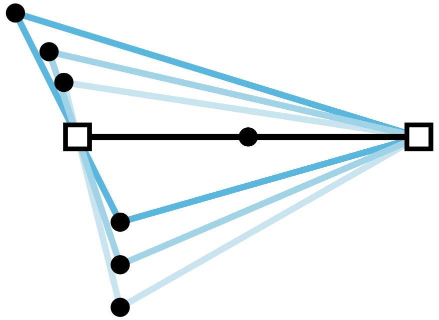
Observation: Dujmović, Eppstein, Suderman, Wood '07

A banana with k parallel paths of length two has segment number $\lfloor 3k/2 \rfloor$.



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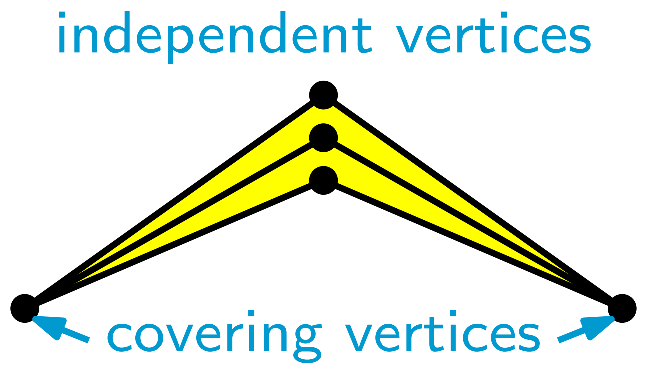

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tree where each edge is replaced by a banana.

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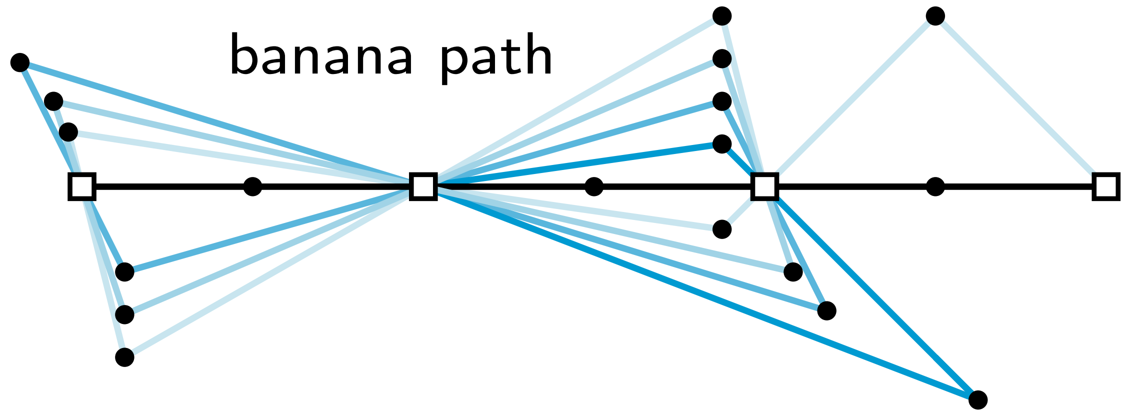

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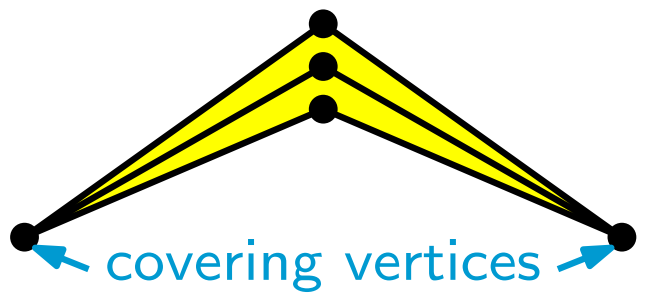

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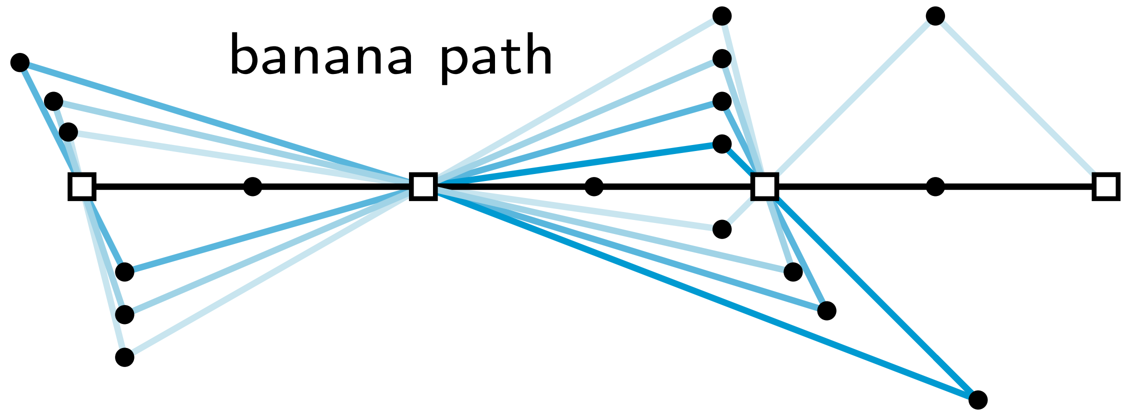

Theorem:

The segment number of a banana tree can be determined in linear time.

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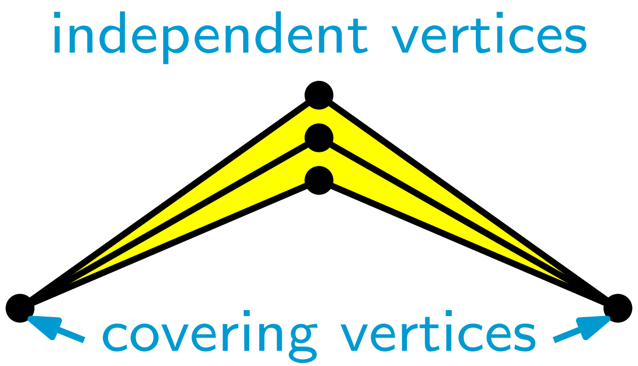

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banana tree (cycle) (Scott/Seymour 2020)
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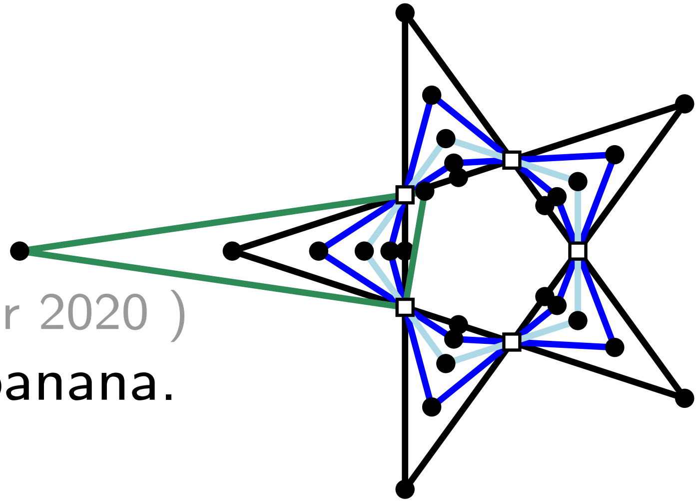
(simple cycle)



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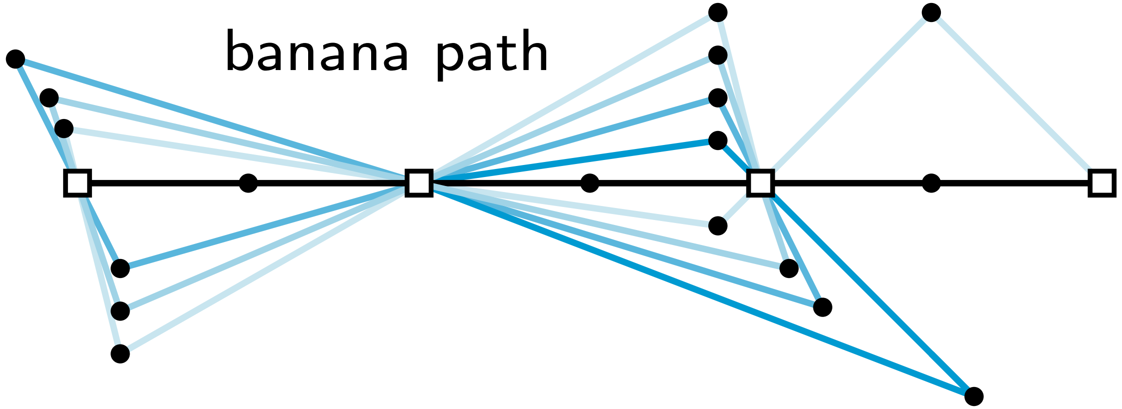

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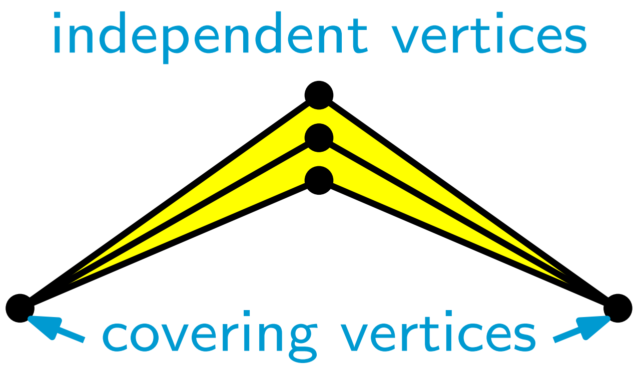

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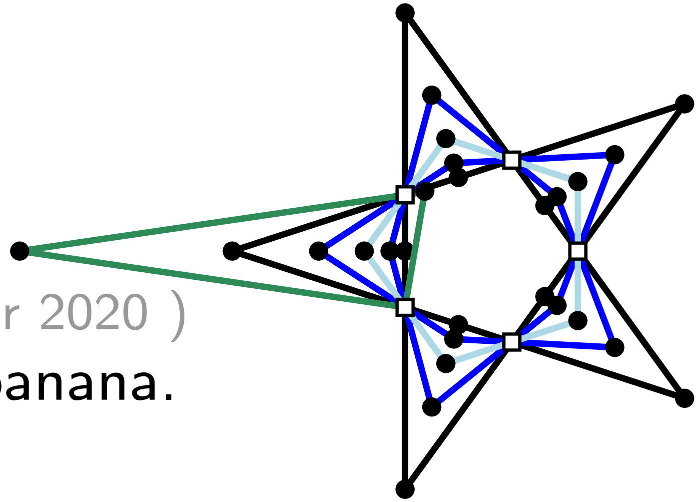

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Theorem:

The segment number of a banana cycle
of length at least five and with at least two independent vertices per banana
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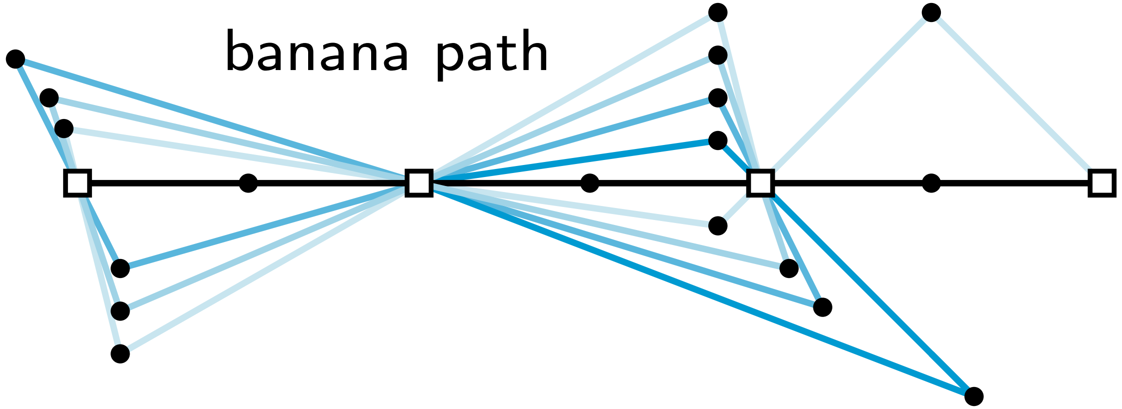

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Related Work

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Bounds for various graph classes, e.g.,

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LINE COVER NUMBER is in FPT wrt. the natural parameter [CFLRVW-JGAA'23]

Recall: decision problem with input x, parameter k is fixed-parameter tractable (FPT) if solvable with run time $\mathcal{O}(f(k)|x|^c)$, c constant, f computable

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Renegar's Decision Algorithm (Renegar, 1992)

Given an existential first-order formula about the reals

$$\exists x_1 \ldots x_m \; \Phi(x_1, \ldots, x_m)$$

(Φ: Boolean combination of equalities and inequalities of polynomials over $\mathbb{Q}$)
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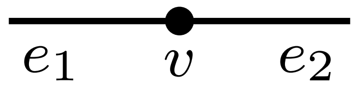
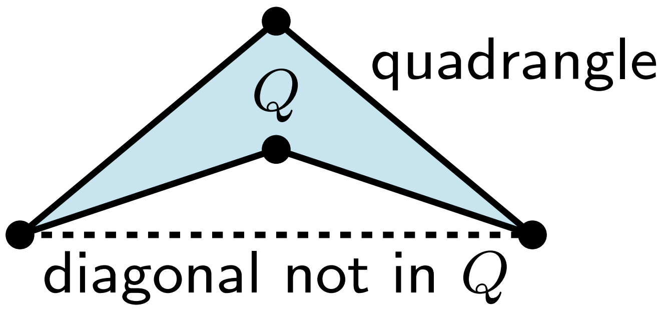
It can be expressed as an existential first-order formula about the reals whether there is a set of points in the plane

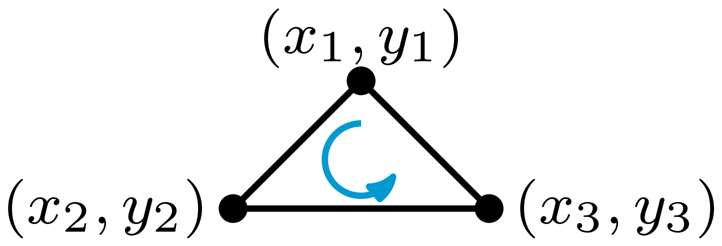
- that is a straight-line planar drawing of a plane graph, (CFLRVW-JGAA'23)
- given pairs of edges are aligned
- given quadrangles are not convex





$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0$$

iff



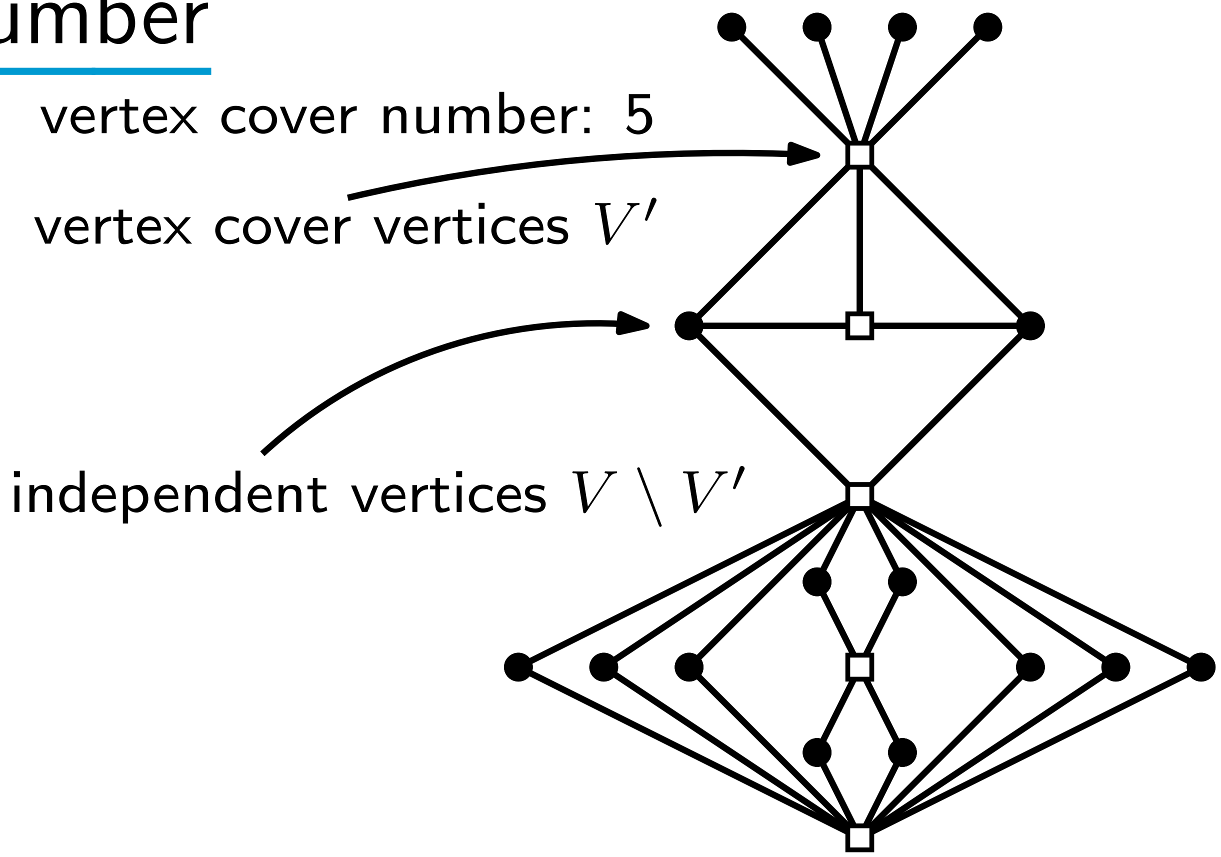
(=0 iff collinear)

$\rightsquigarrow$ $\mathcal{O}(|V|^{\mathcal{O}(|V|)})$ algorithm for Segment Number

SEGMENT NUMBER by Vertex Cover Number

vertex cover of a graph $G = (V, E)$: set $V' \subseteq V$ s.t. $e \cap V' \neq \emptyset$ for each $e \in E$

vertex cover number of a graph: size of its smallest vertex cover



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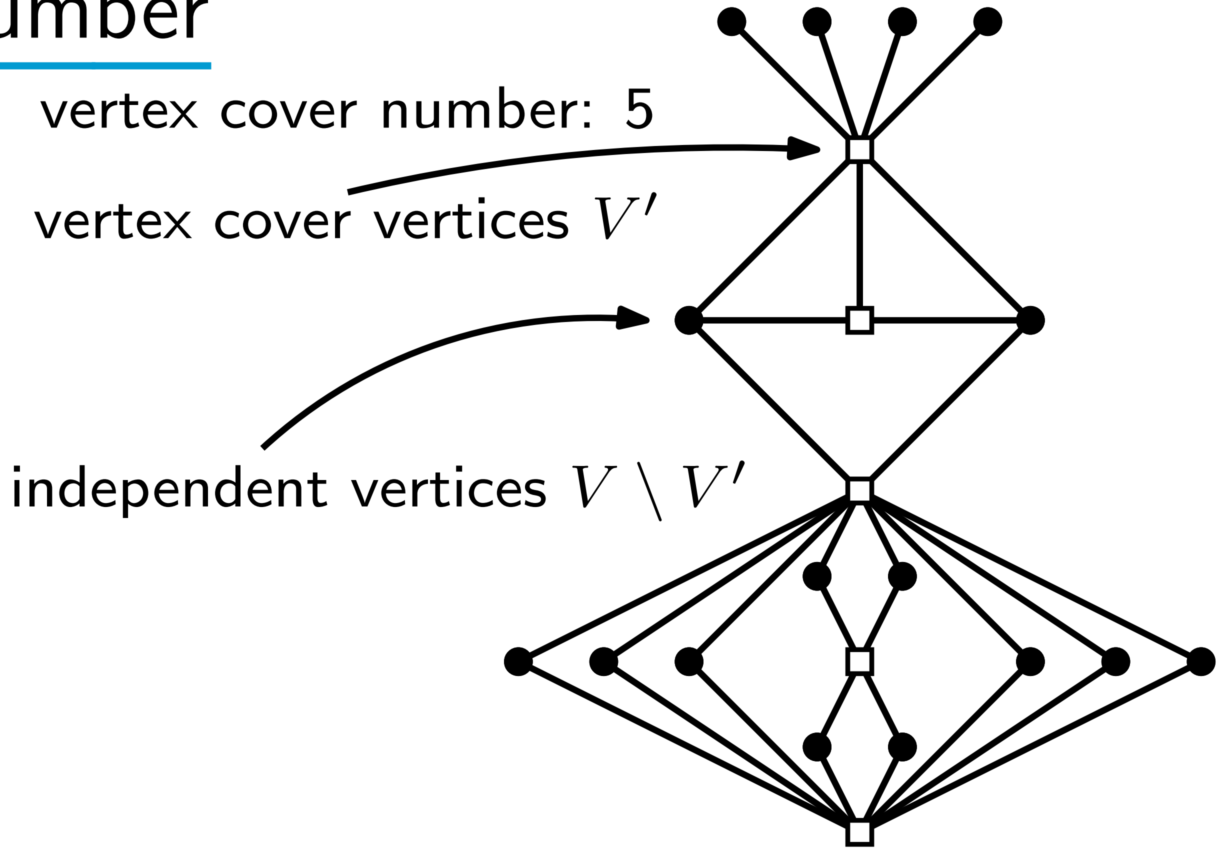

Segment Number by Vertex Cover Number
Input: planar graph $G = (V, E)$, integer s
Parameter: vertex cover number k of G
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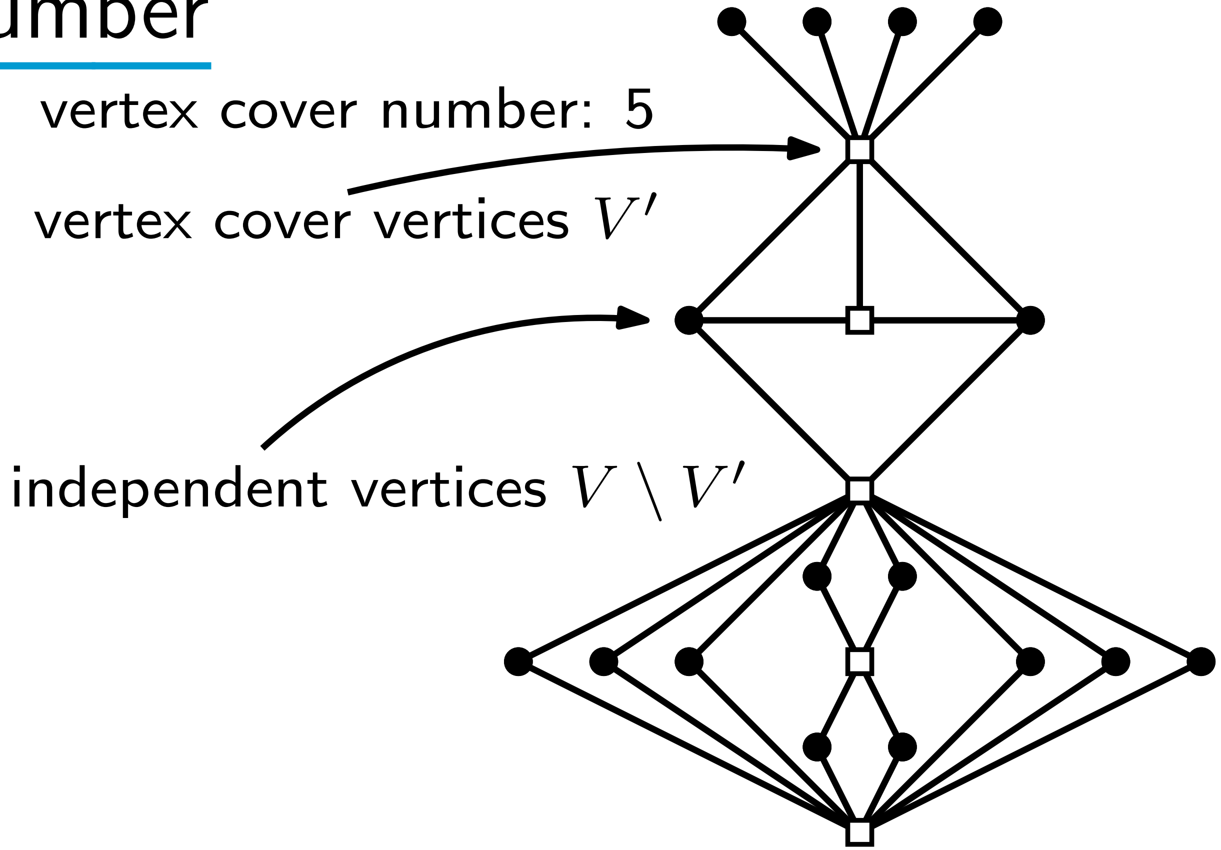

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Overview of the Approach for **computing** the segment number:

1. Remove some vertices of degree one and two $\rightsquigarrow$ $\mathcal{O}(2^k)$ vertices
2. Iterate over all possible embeddings and alignments $\rightsquigarrow$ number of choices is a function in k
3. Use Renegar to test for realizability $\rightsquigarrow$ $2^{\mathcal{O}(k2^k)}$ time per choice
4. Reinsert the missing vertices optimally via an ILP $\rightsquigarrow$ $2^{\mathcal{O}(k2^k)}$ time per choice

→ Take the best

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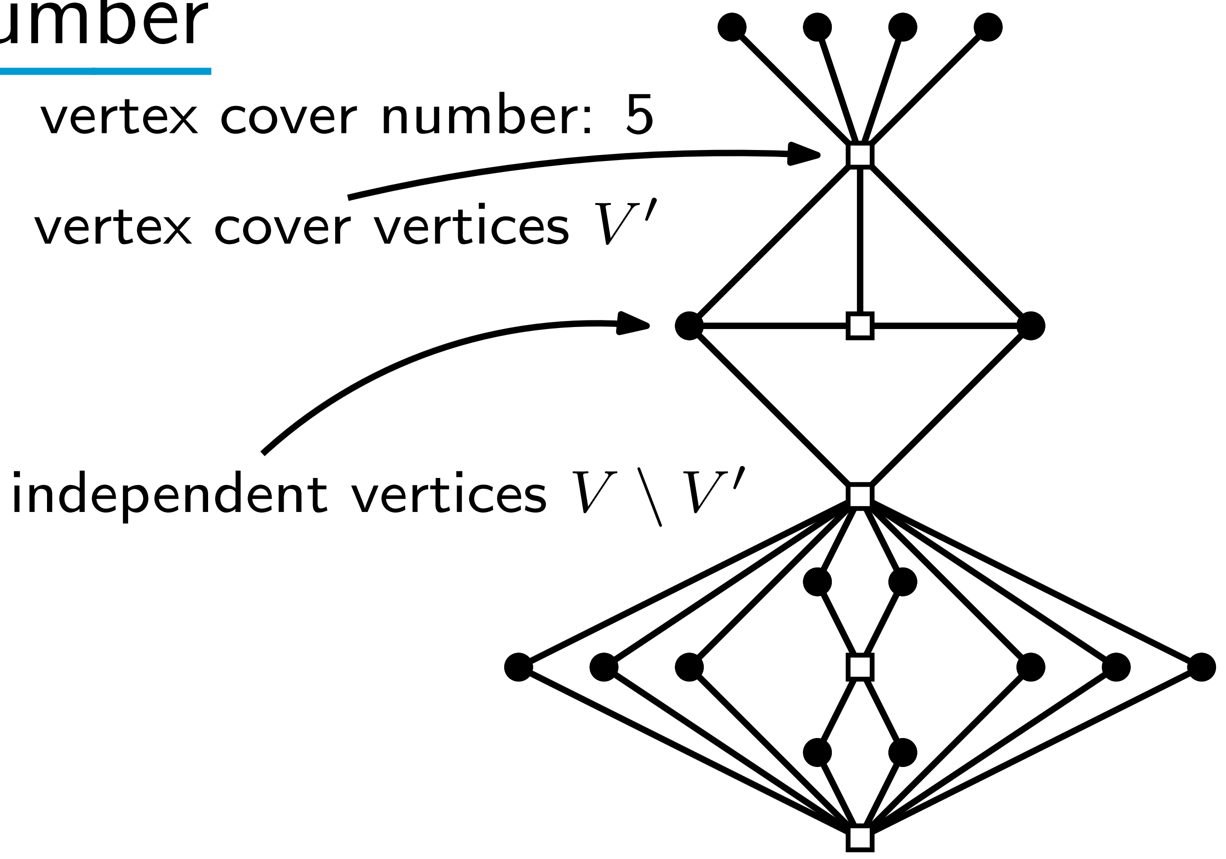

$\rightsquigarrow \mathcal{O}(2^k)$ vertices

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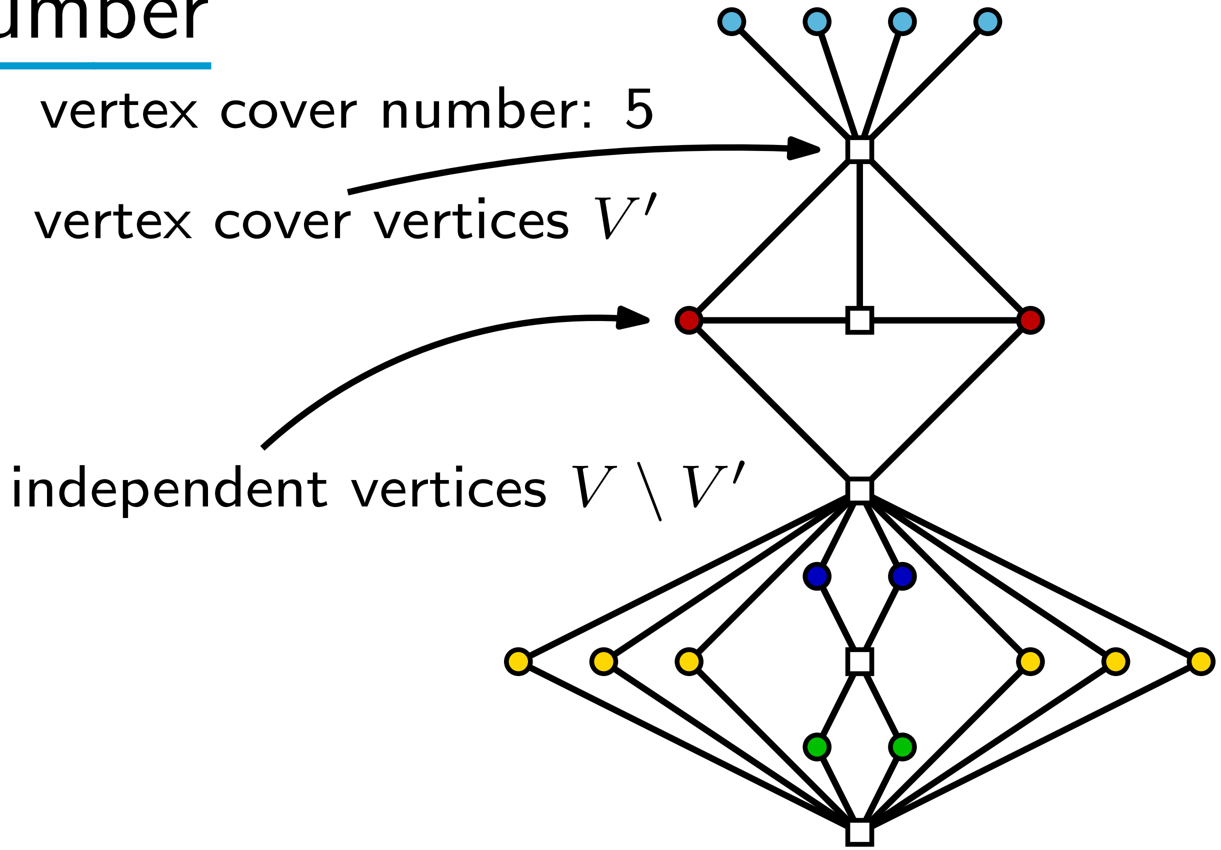

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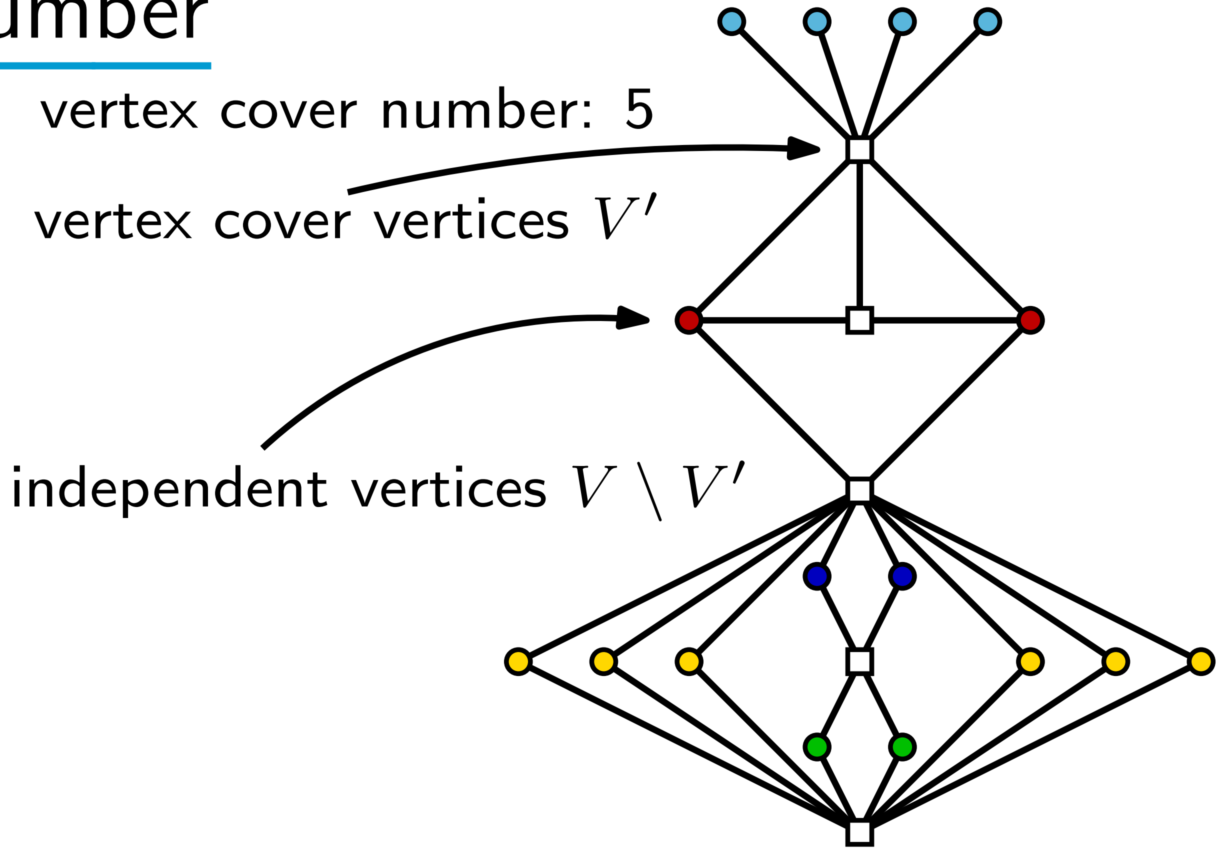

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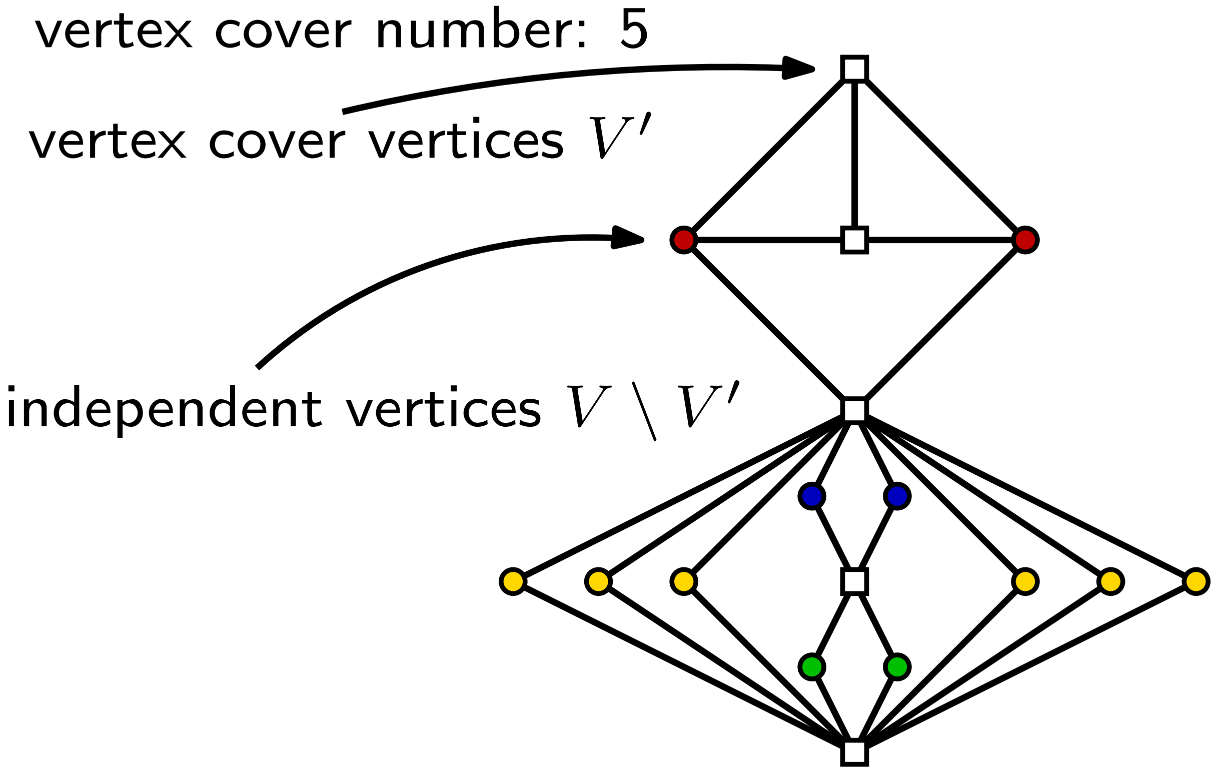

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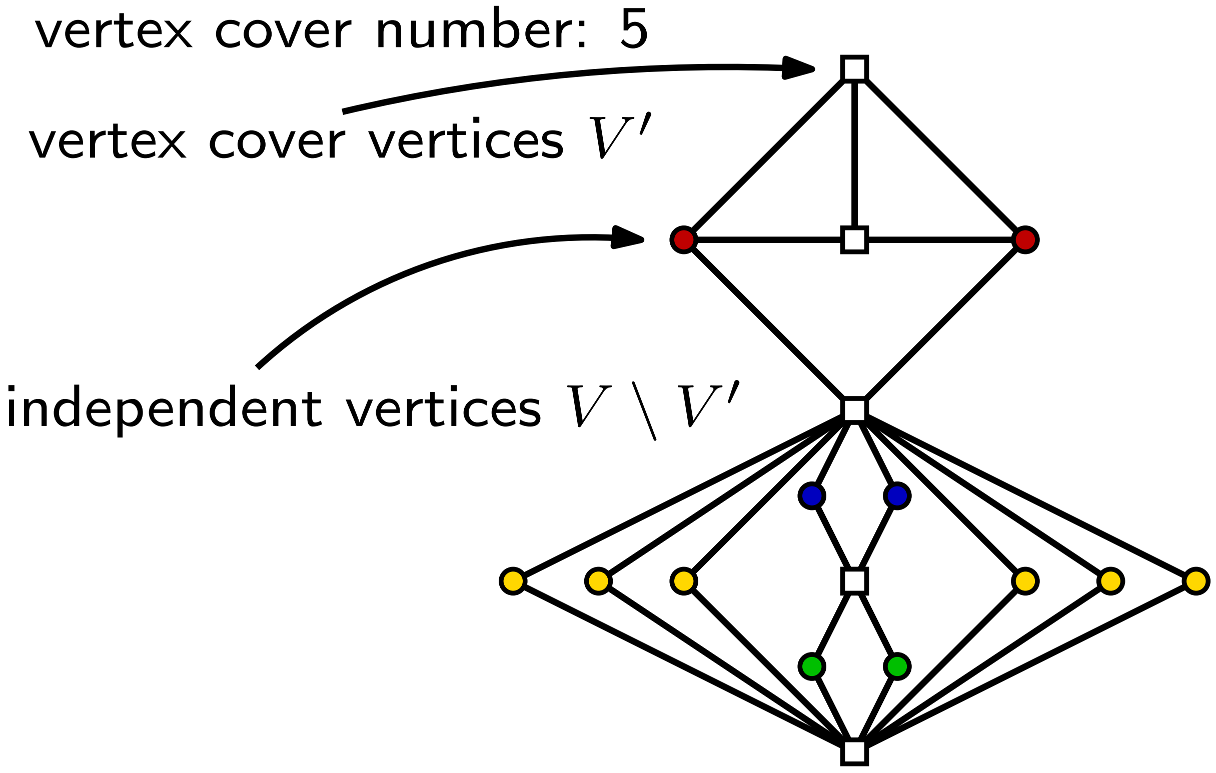
$\rightsquigarrow$ one per contiguous 2-class



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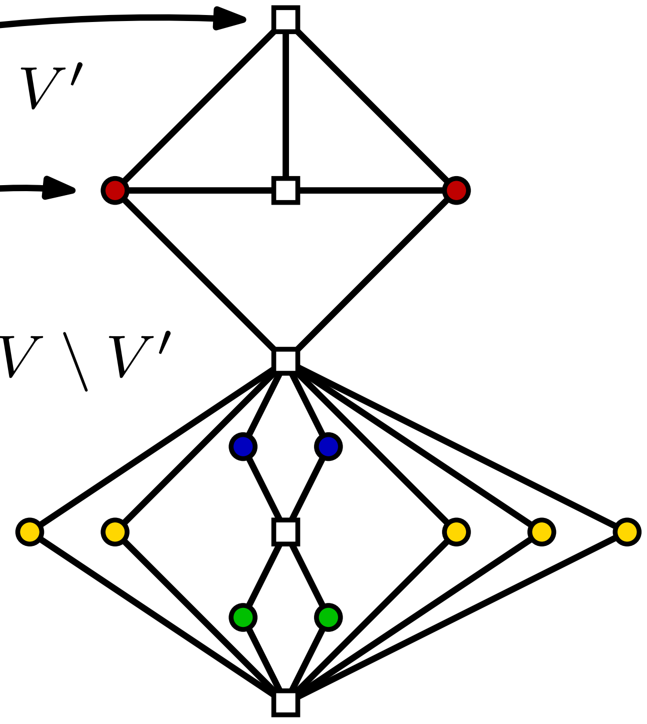
vertex cover number: 5

vertex cover vertices V'

independent vertices $V \setminus V'$



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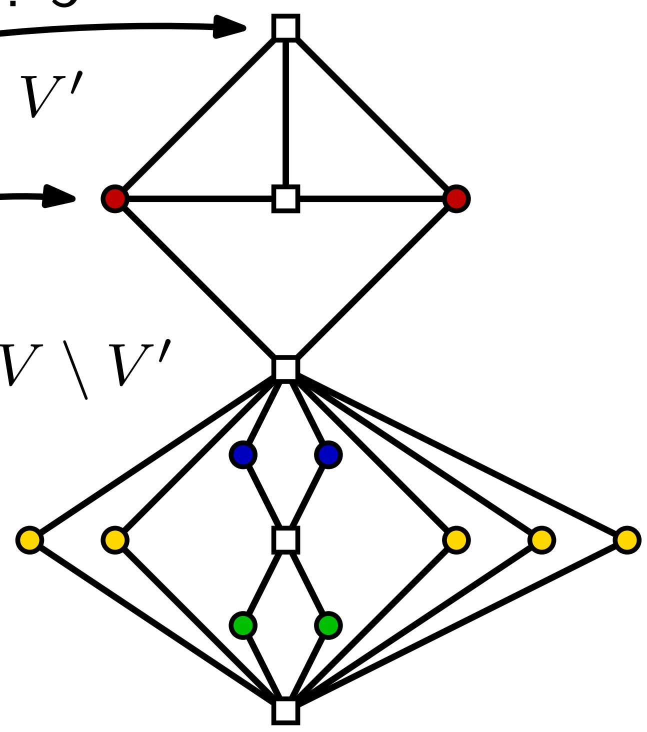

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c) Each j-class, $j > 2$ contains at most two vertices otherwise there would be a $K_{3,3}$

$2 \cdot \sum_{j=3}^{k} \binom{k}{j} \in \mathcal{O}(2^k)$

$\rightsquigarrow \mathcal{O}(2^k)$ vertices

Segment Number by Vertex Cover Number

1. Remove some vertices of degree one and two

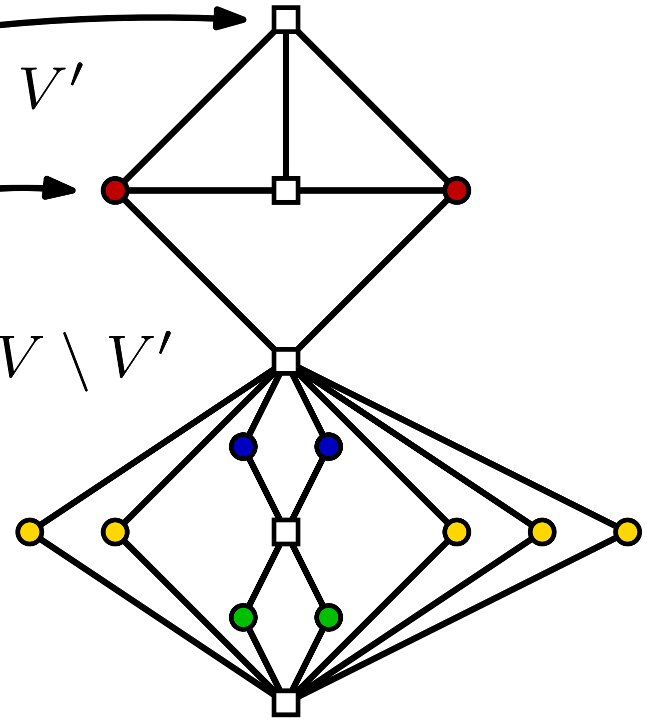
Two independent vertices v, v' are equivalent iff adjacent to the same vertices in V'

j-class: equivalence class where each vertex is adjacent to exactly j vertices.

vertex cover number: 5

vertex cover vertices V'

independent vertices $V \setminus V'$



a) Remove all vertices of degree 1 (1-classes)

b) For each 2-class, maintain at most k vertices $\rightsquigarrow$ one per contiguous 2-class — $k\binom{k}{2}$

c) Each j-class, $j > 2$ contains at most two vertices otherwise there would be a $K_{3,3}$ — $2 \cdot \sum_{j=3}^{k} \binom{k}{j} \in \mathcal{O}(2^k)$

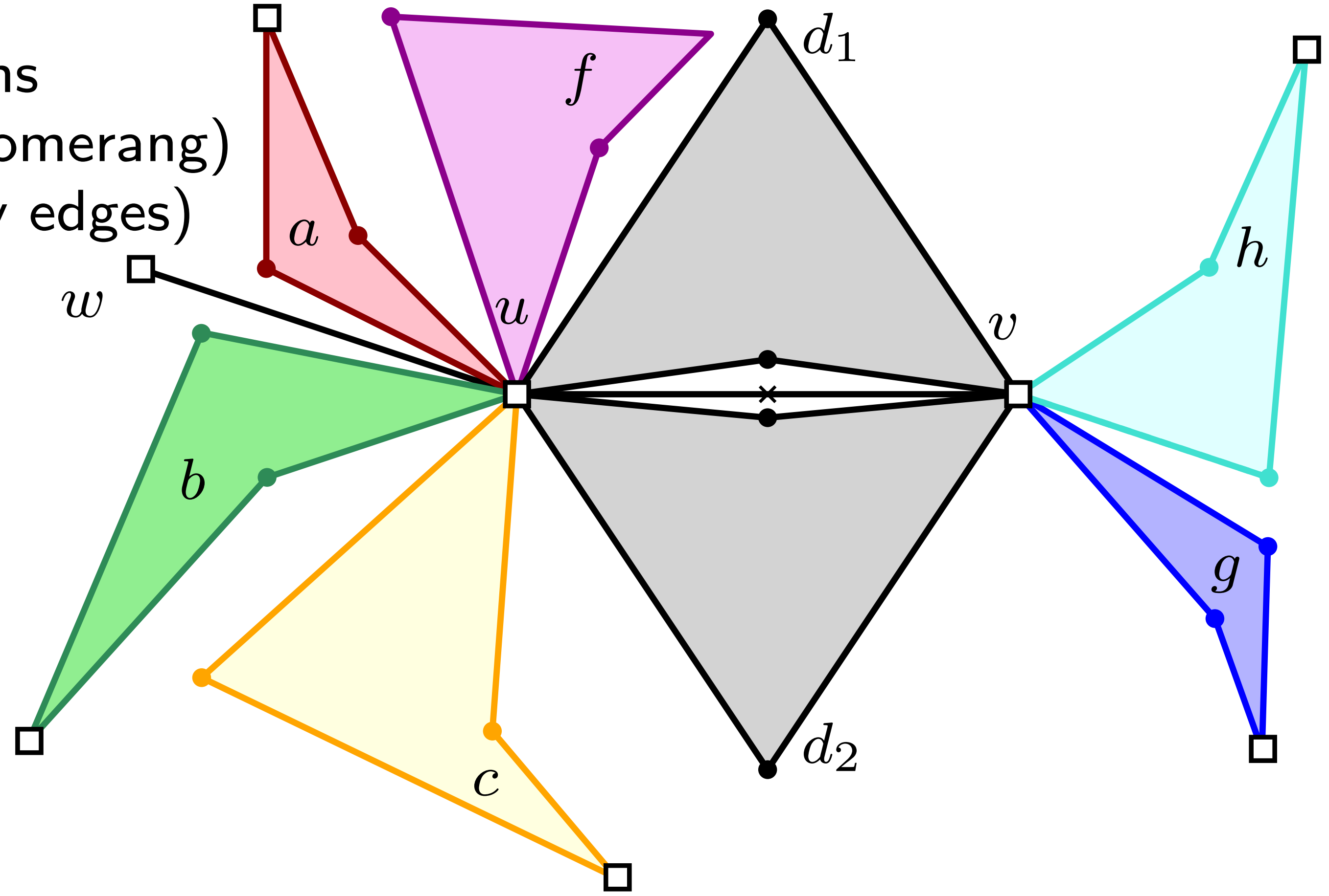
d) Vertex cover — k

$\rightsquigarrow \mathcal{O}(2^k)$ vertices

SEGMENT NUMBER by Vertex Cover Number

2. Iterate over all possible embeddings and alignments

a) each contiguous 2-class is represented by 4 paths
which must form a non-convex quadrangle (boomerang)
(alignments at independent vertices represented by edges)



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a) each contiguous 2-class is represented by 4 paths
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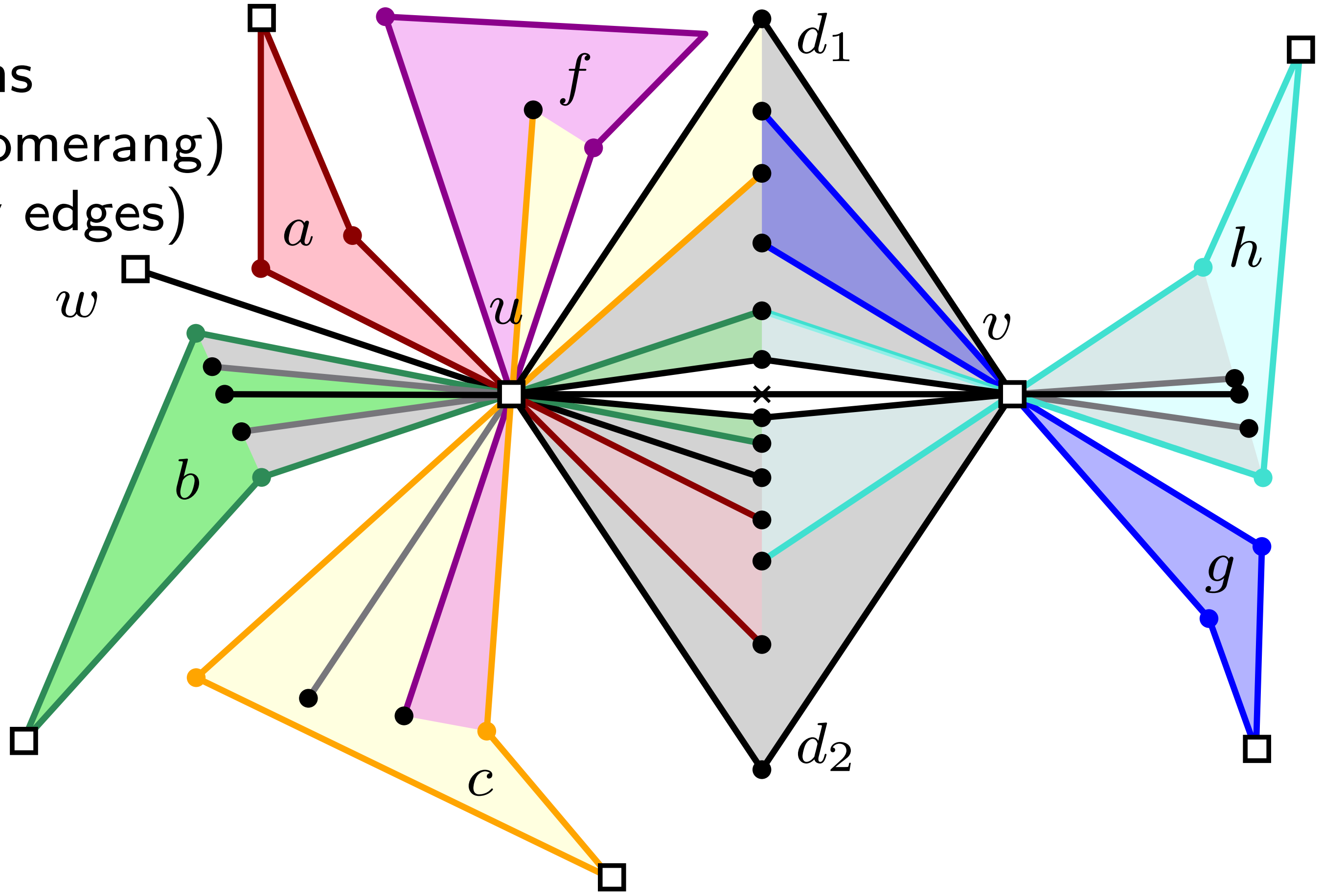
b) further subdivide boomerangs,
according to the choice of the alignments

still $\mathcal{O}(2^k)$ vertices



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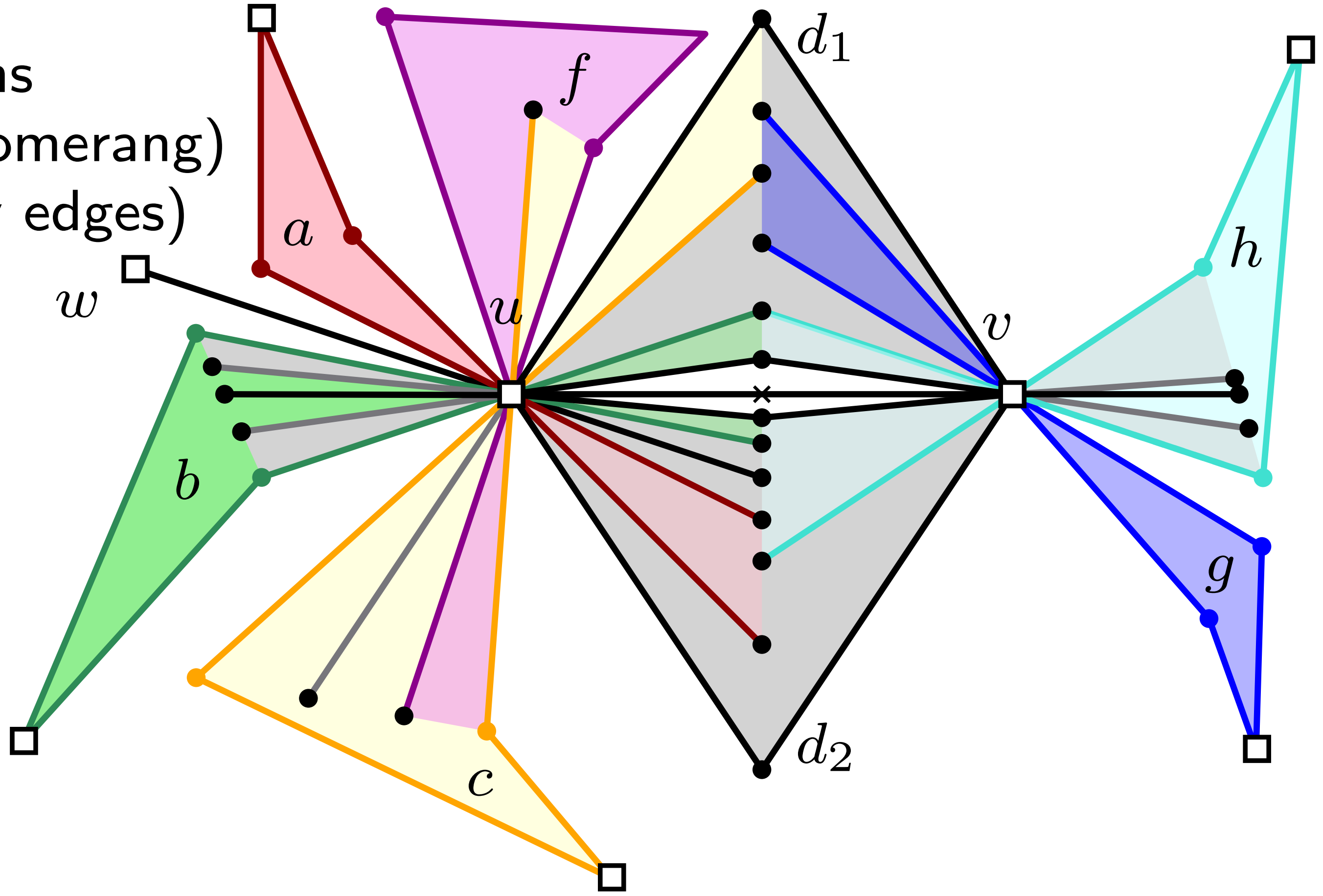
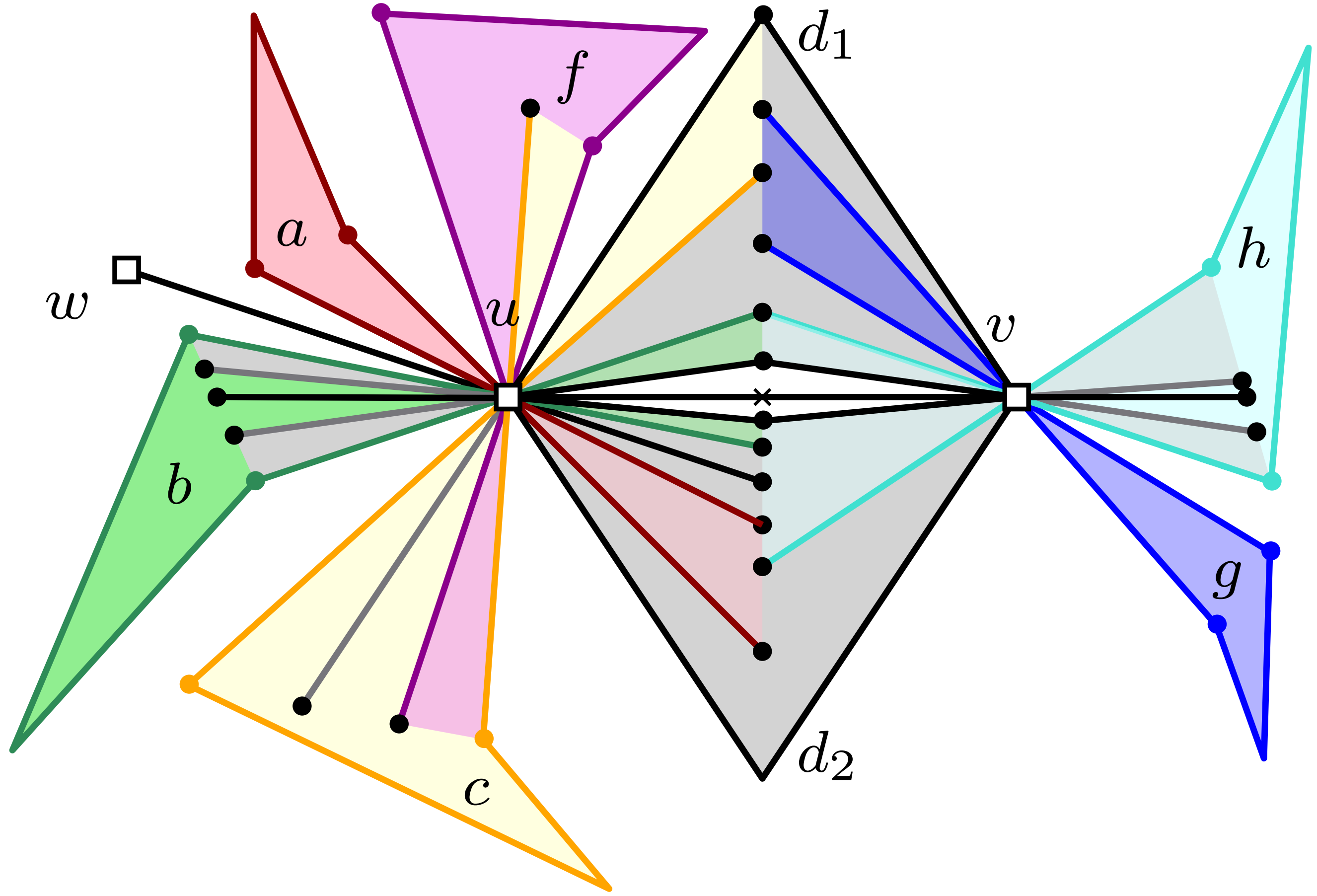


3. Use Renegar to test in $2^{\mathcal{O}(k2^k)}$ time for realizability

if the answer is yes then . . .

Segment Number by Vertex Cover Number

4. Reinsert the missing vertices optimally via an ILP



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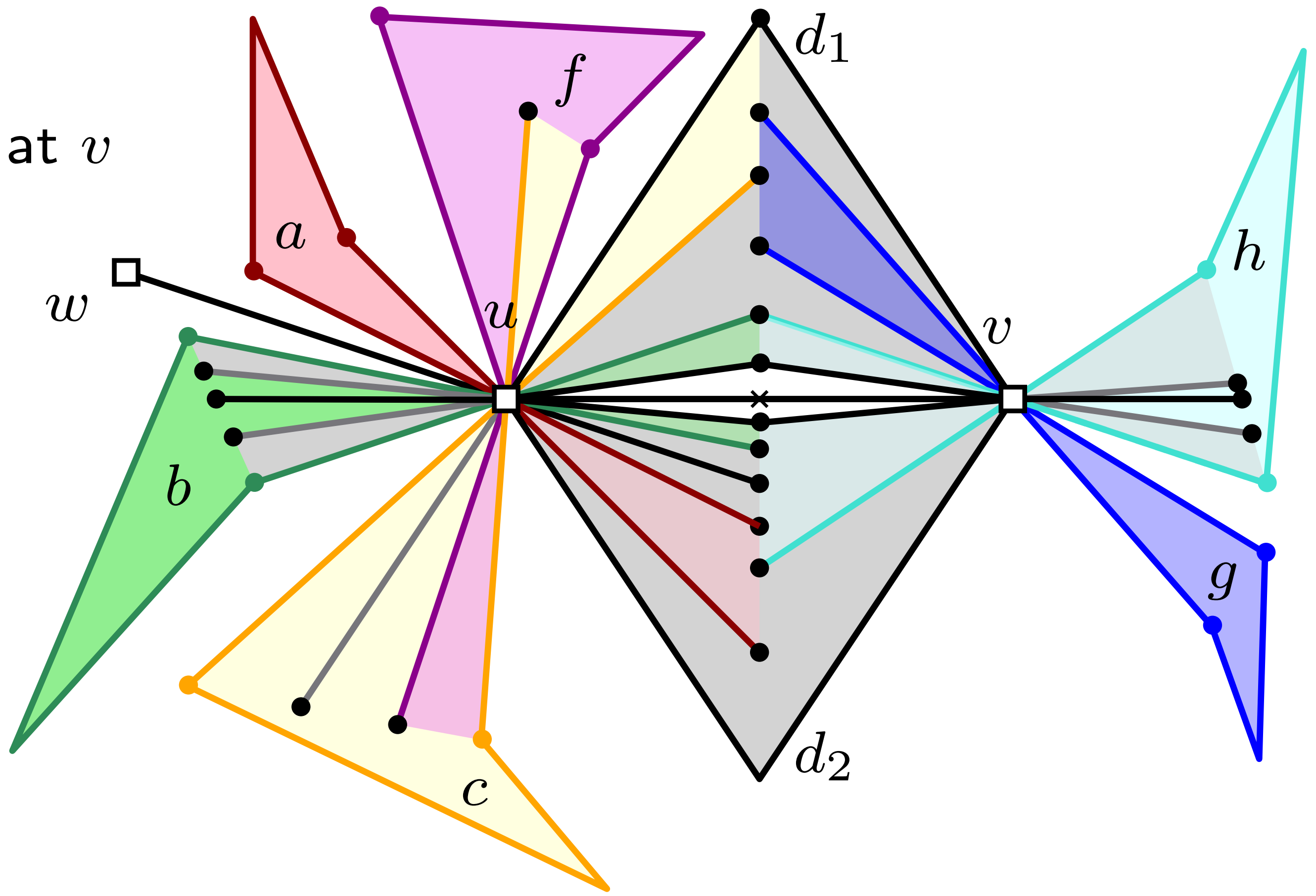
number of edges in b and d that should be aligned at v



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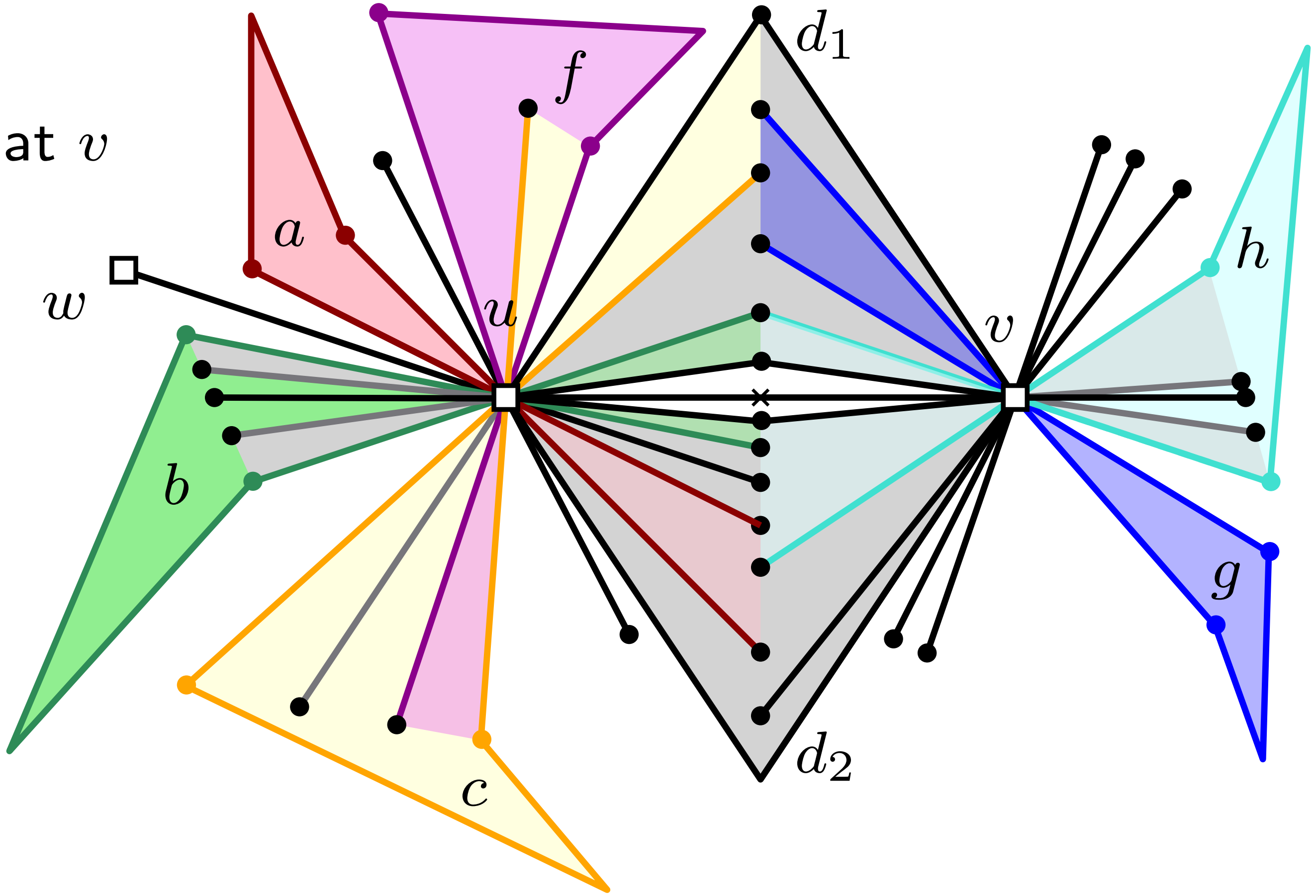

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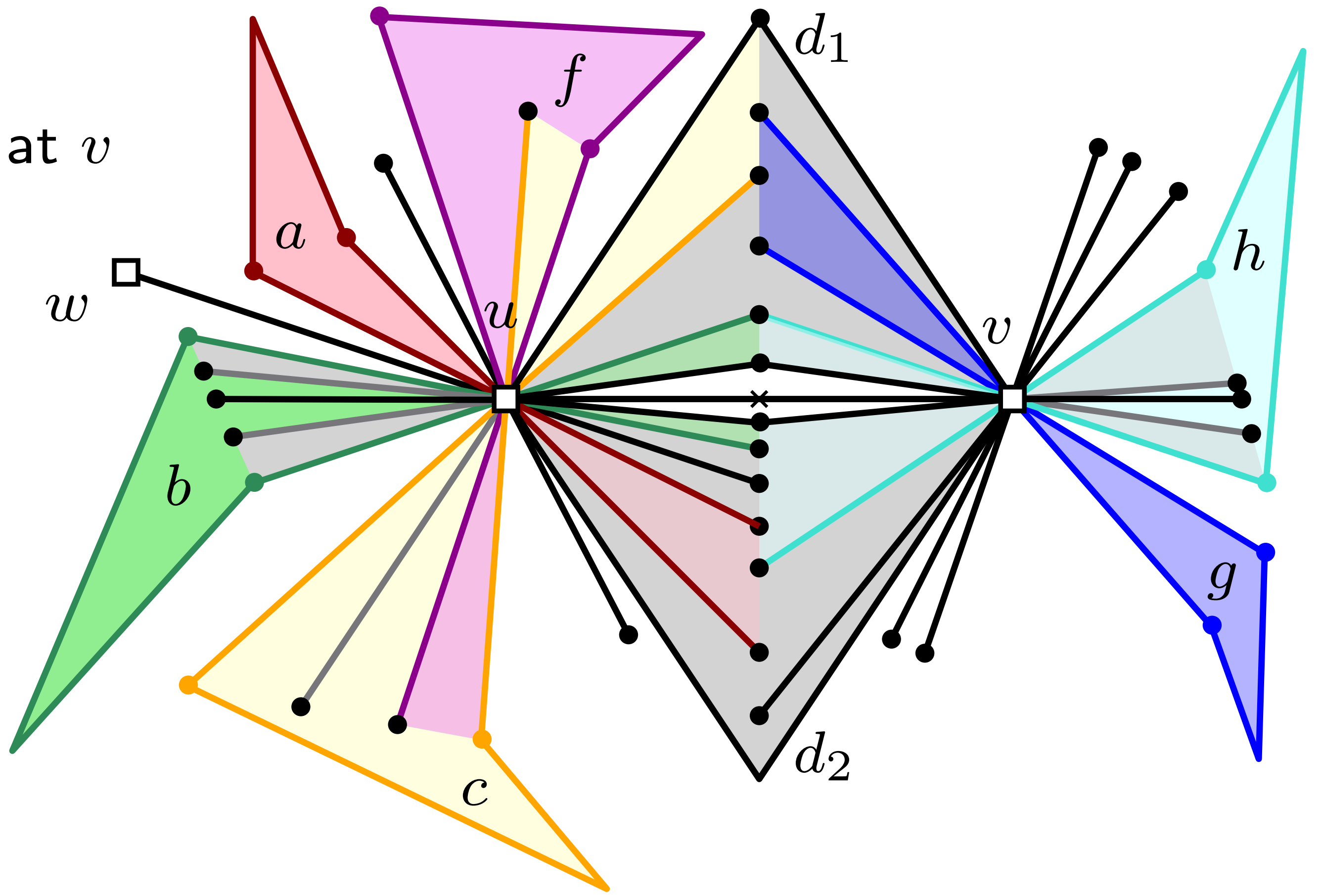
maximize $x + y$



make sure that total number of independent vertices per 1- and 2-class is not exceeded

$\mathcal{O}(2^k)$ variables and constraints
$\rightsquigarrow$ can be solved in $2^{\mathcal{O}(k2^k)}$ time

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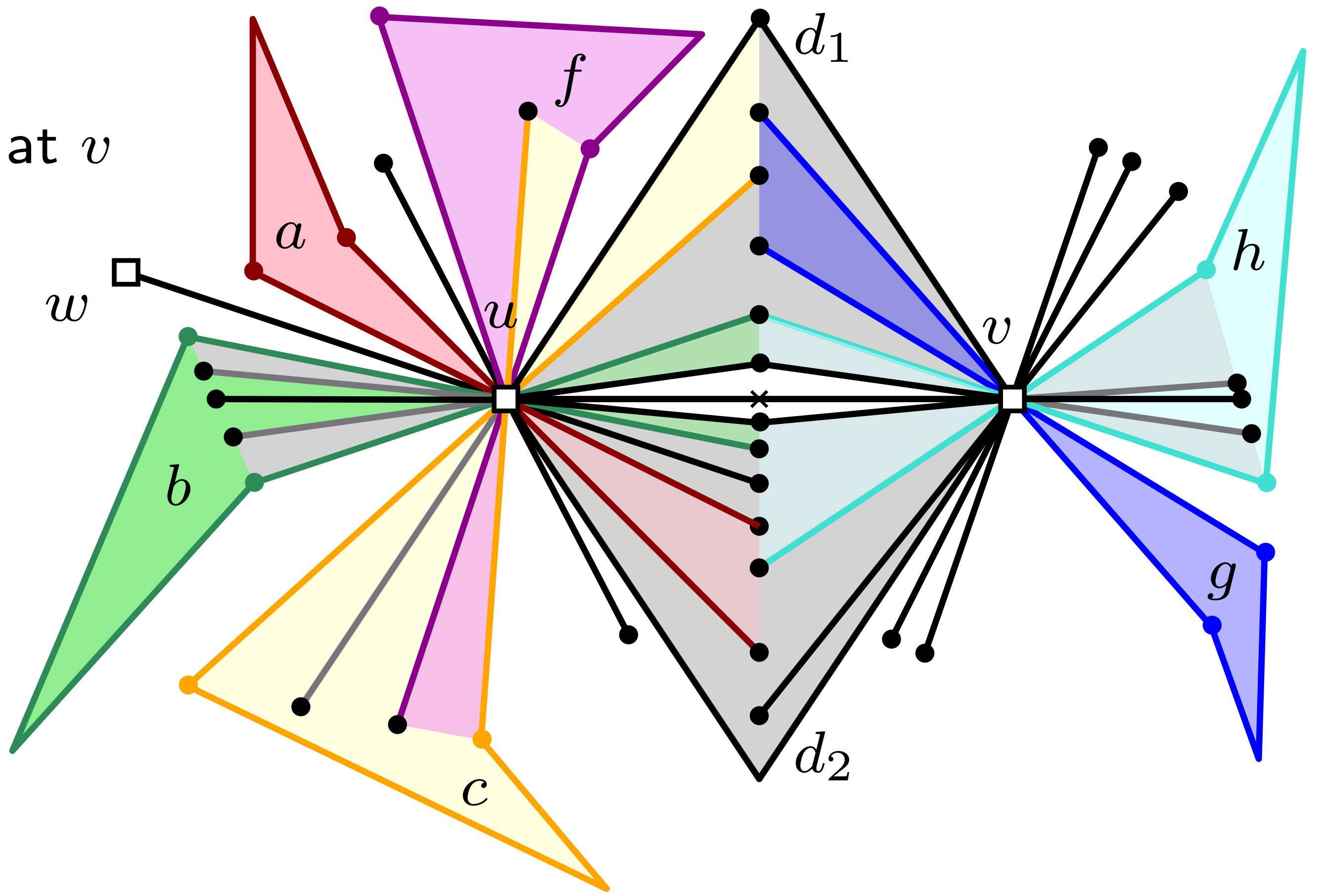
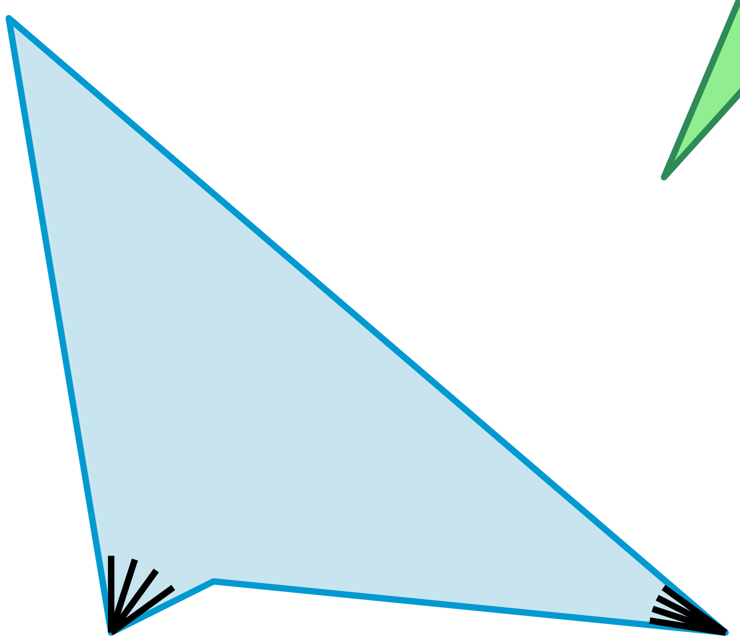


Observe:
Due to the non-convex shape, any given slopes on either sides can be combined s.t. intersection point lies inside boomerang.



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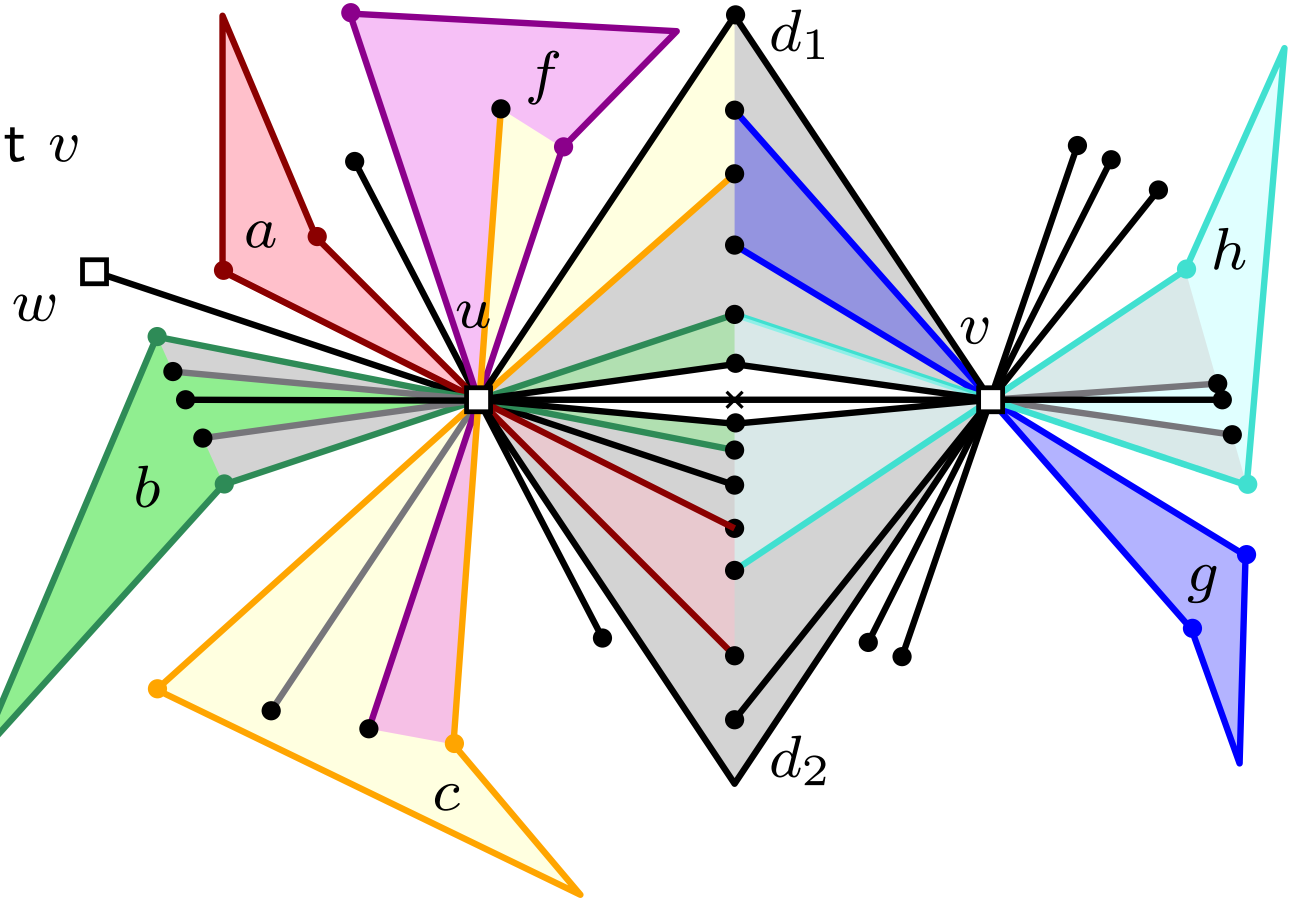

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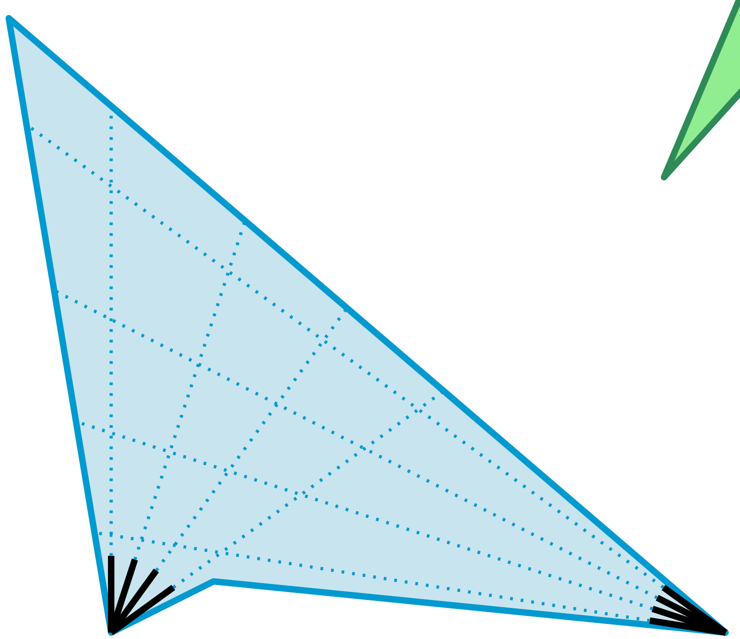

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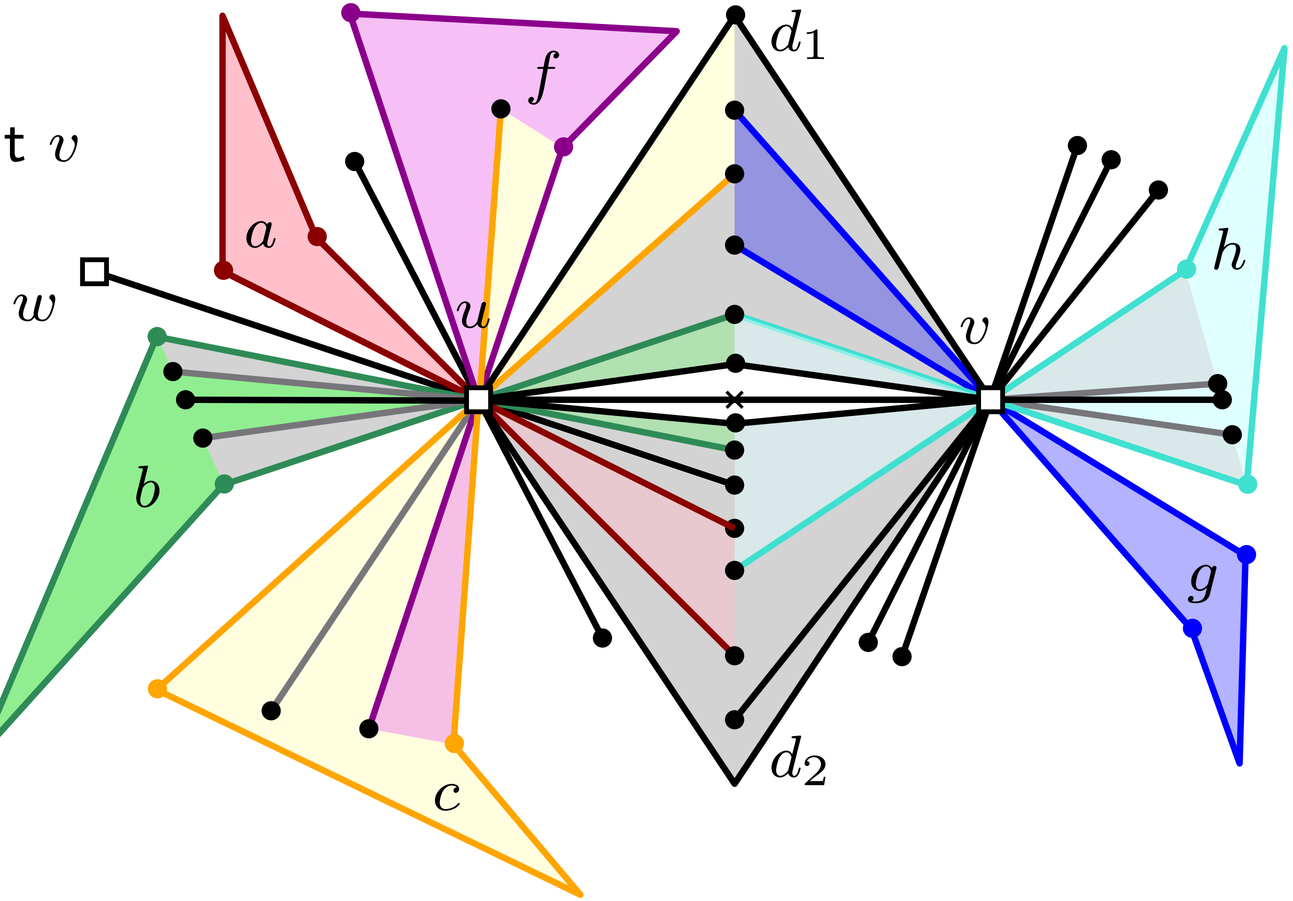

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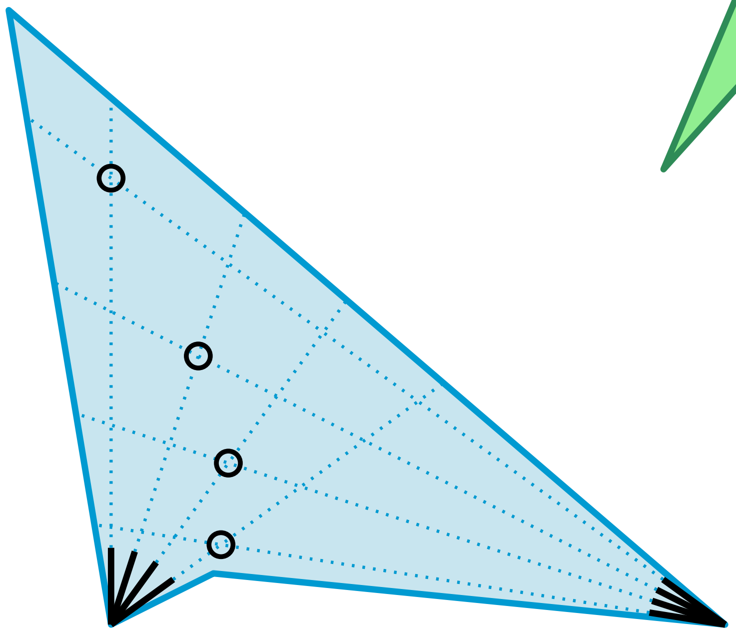

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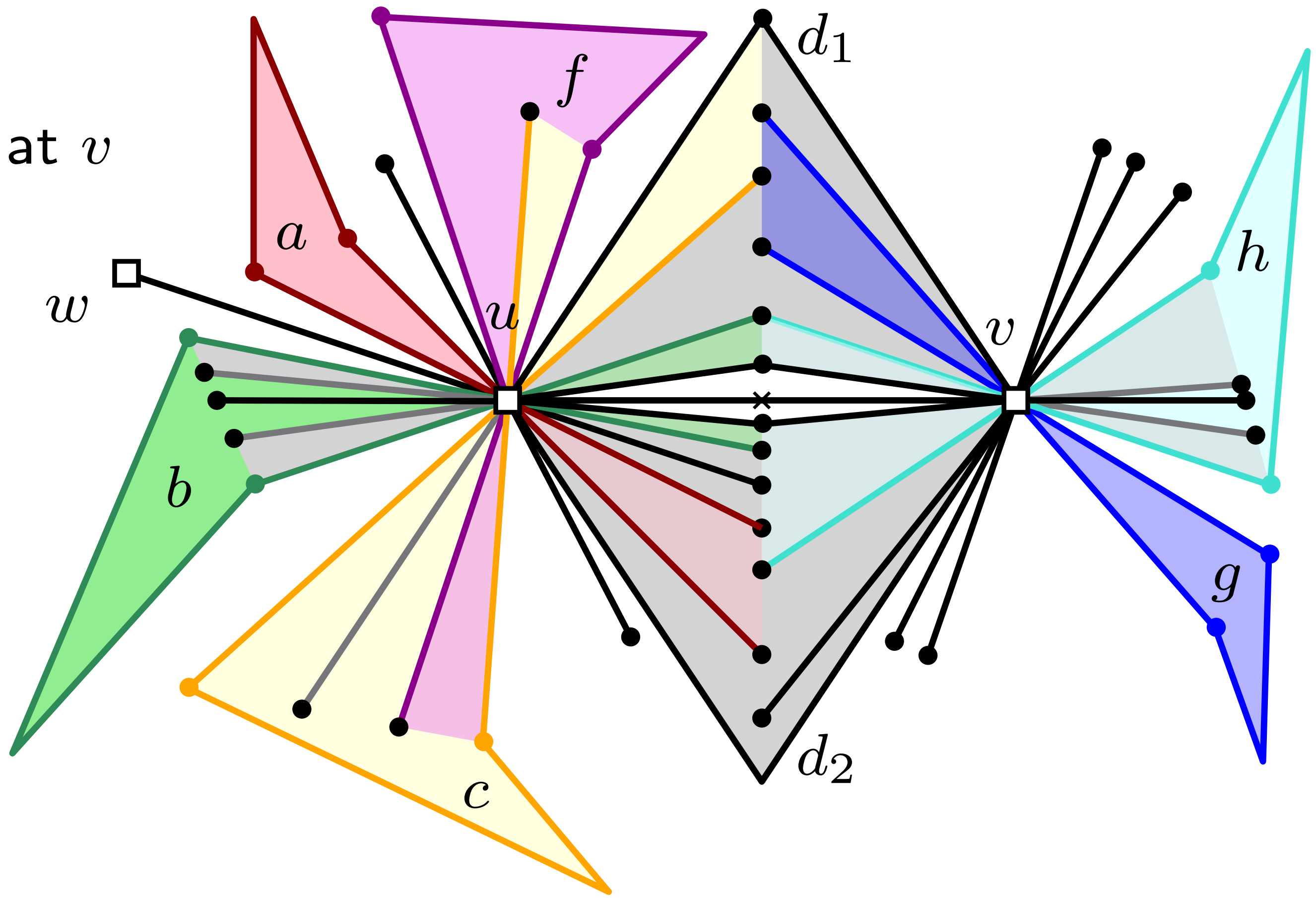

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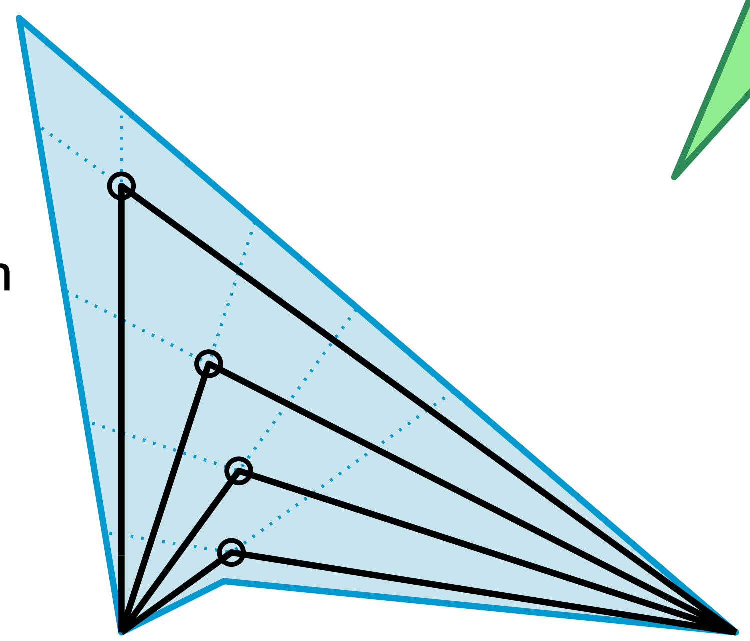

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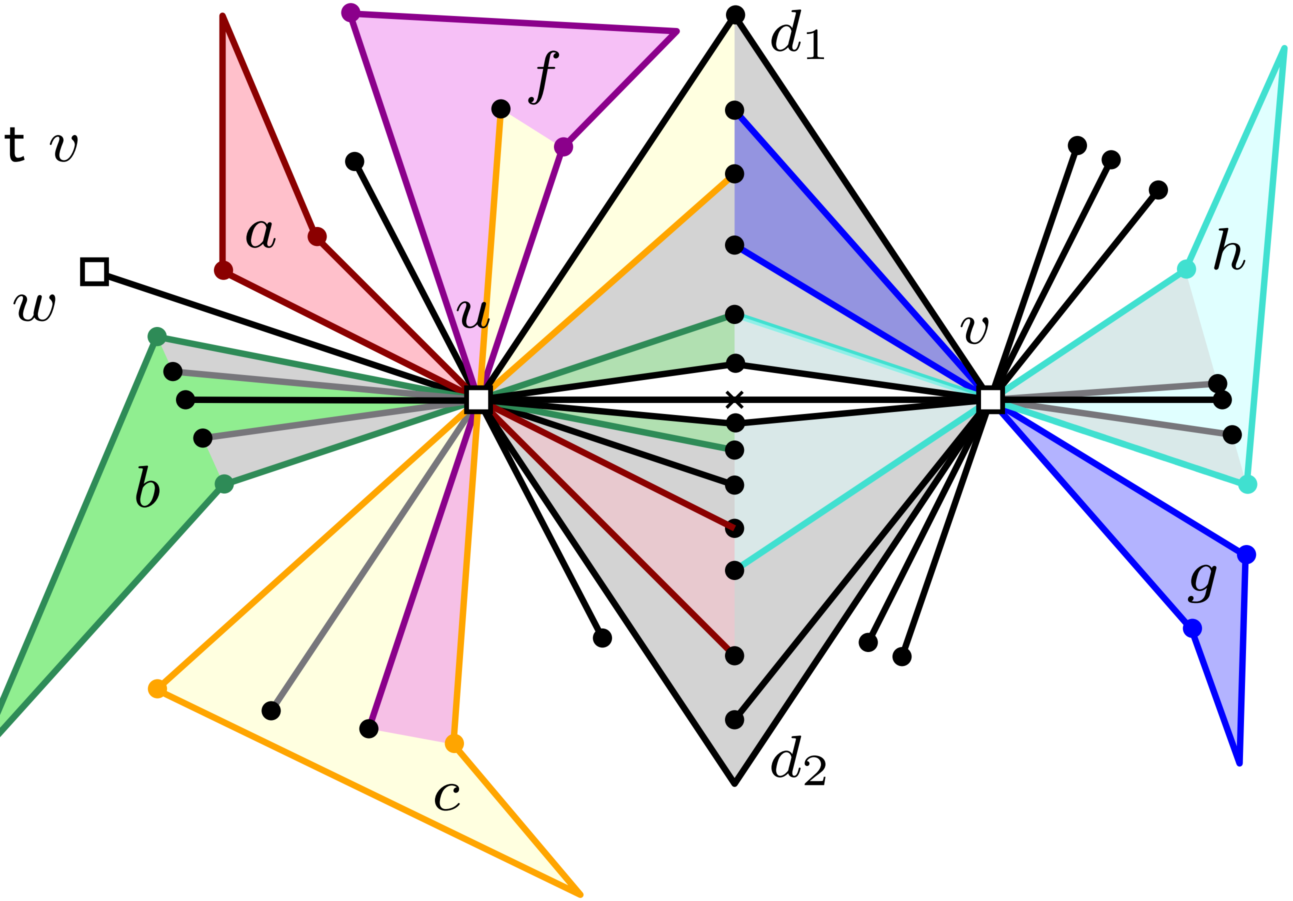

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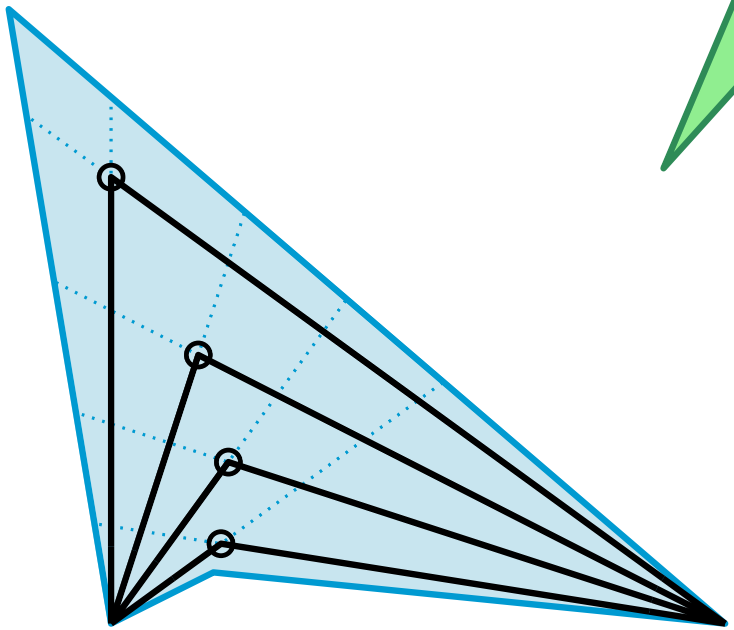

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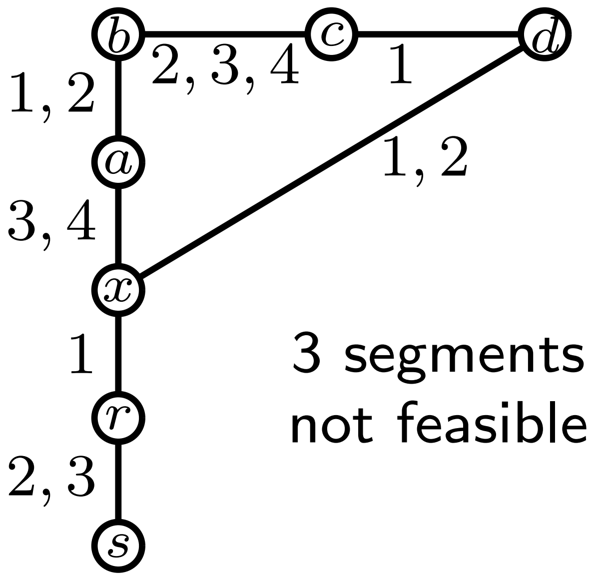
List-Coloring meets Segment Number

List-Incidence Segment Number

Input: planar graph G and, for each $e \in E(G)$, a list $L(e) \subseteq [k]$.
Parameter: An integer k.
Question: Does there exist

- a planar straight-line drawing of G with $\leq k$ segments and
- a labeling $s_1, s_2, \ldots$ of its segments, s.t.
- for every $e \in E(G)$, e is drawn on a segment in $\{s_i : i \in L(e)\}$?



3 segments
not feasible

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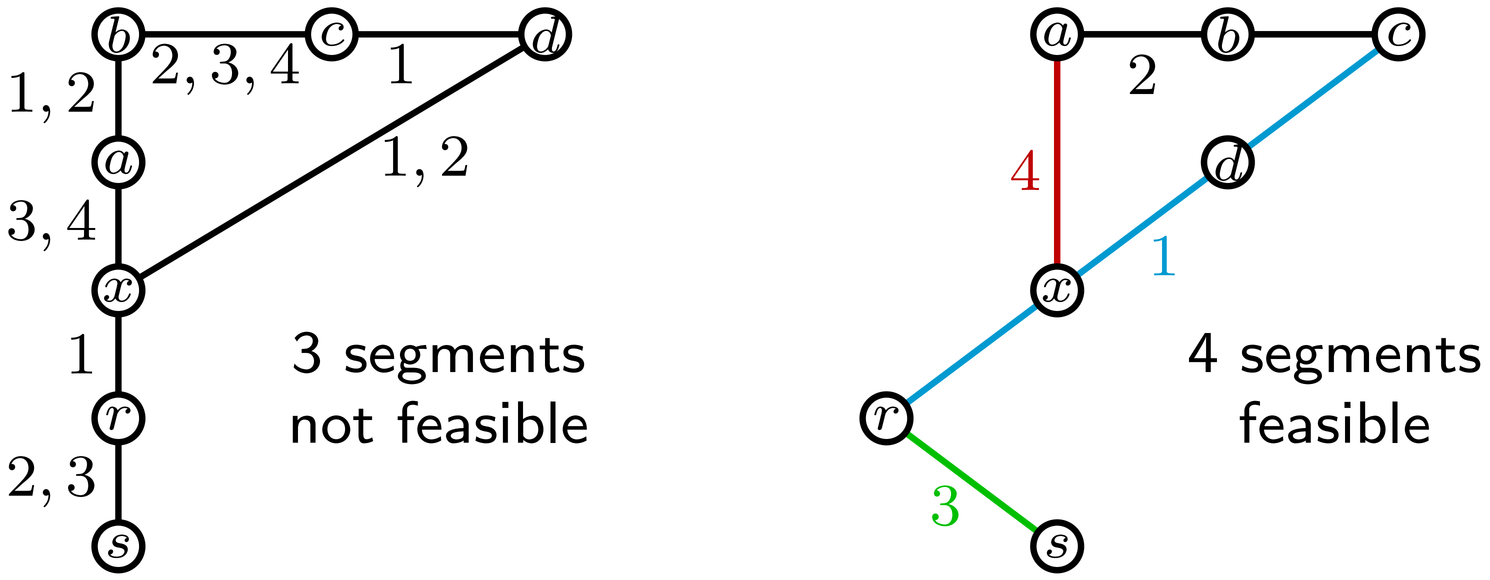

List-Incidence Segment Number is in FPT wrt. Segment Number

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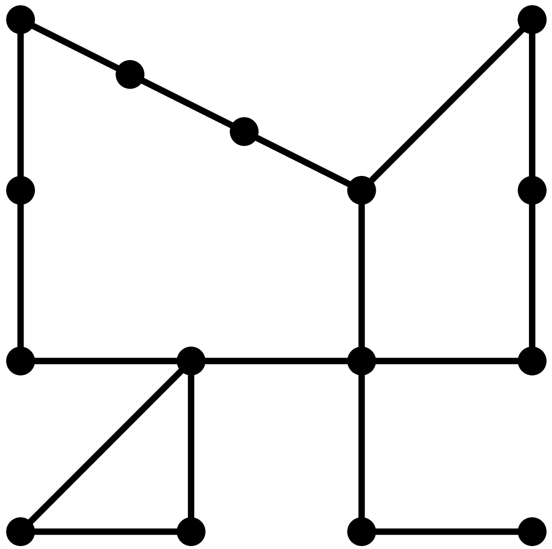

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Split G into

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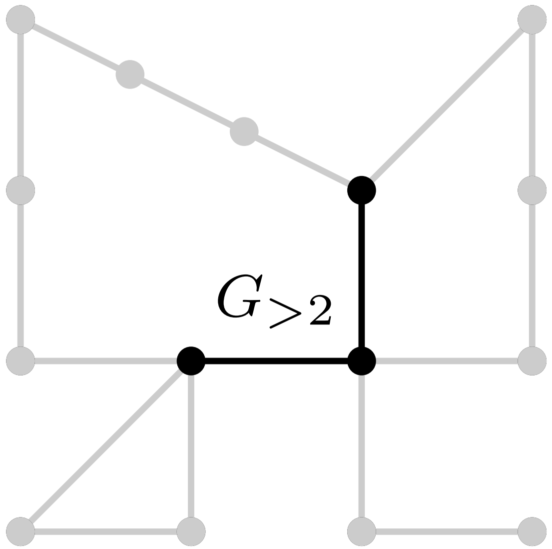
- light paths of degree 2.



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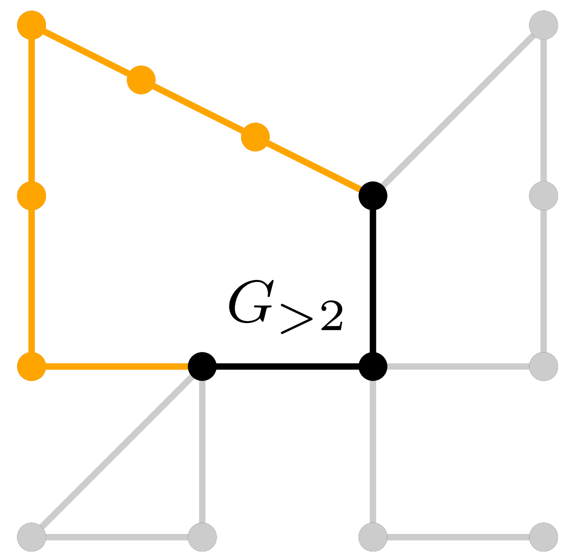

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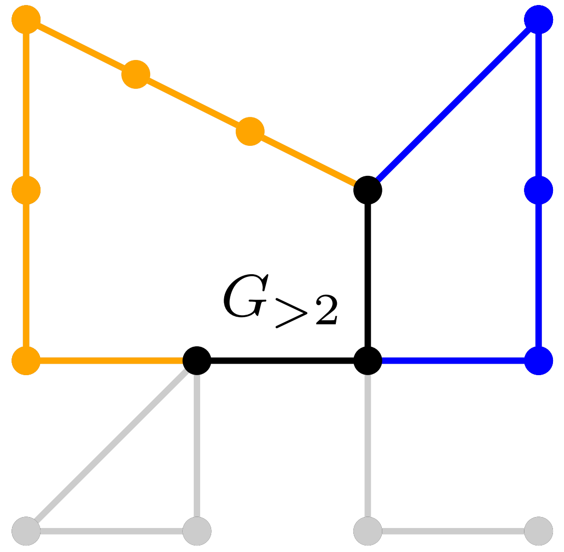

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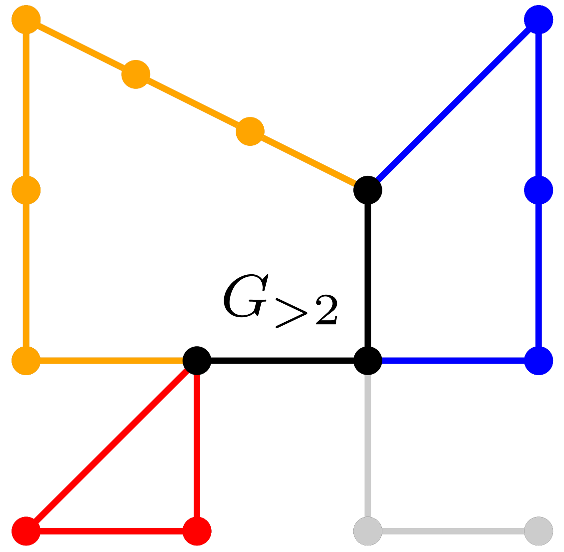

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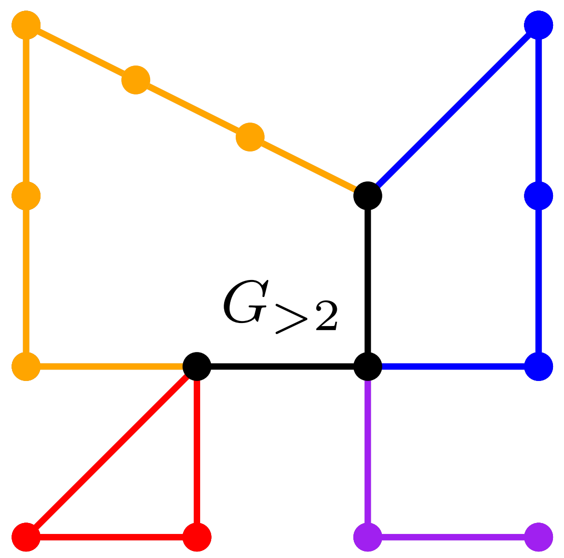

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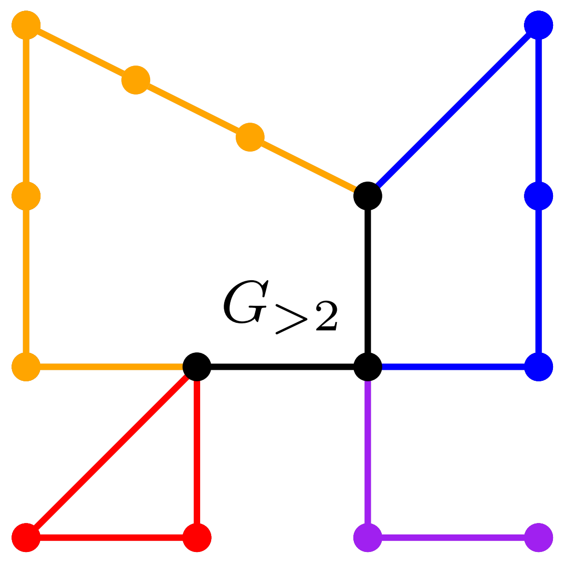
Input: planar graph $G = (V, E)$, integer k, lists $L(e) \subseteq [k]$, $e \in E$

Split G into $|V_{>2}| \leq \binom{k}{2}$

- graph $G_{>2} = (V_{>2}, E_{>2})$ induced by vertices of degree > 2
- light paths of degree 2. at most $2k \cdot \binom{k}{2}$ many



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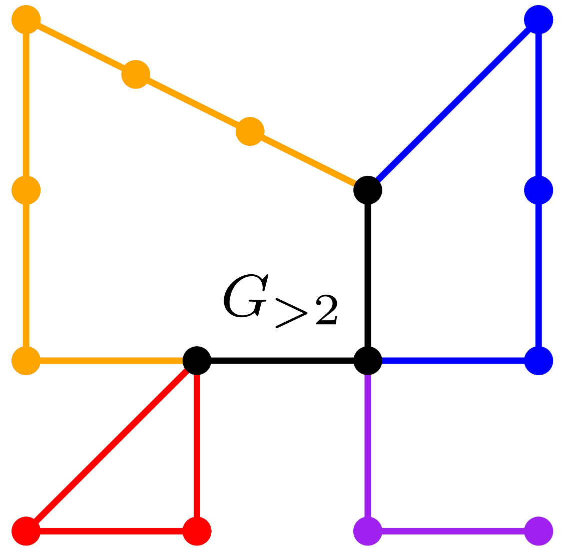
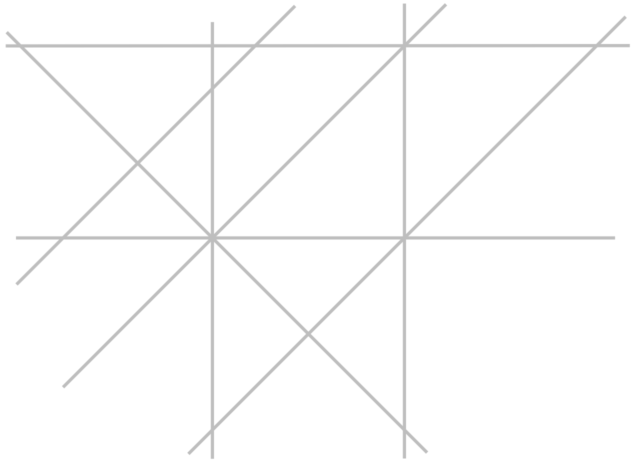
1. For each arrangement of k lines





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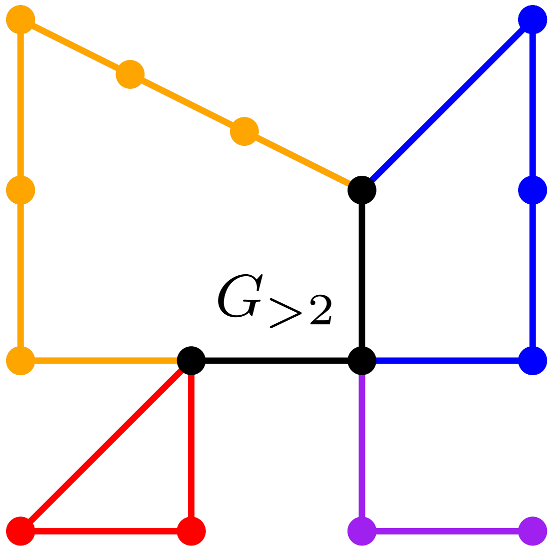
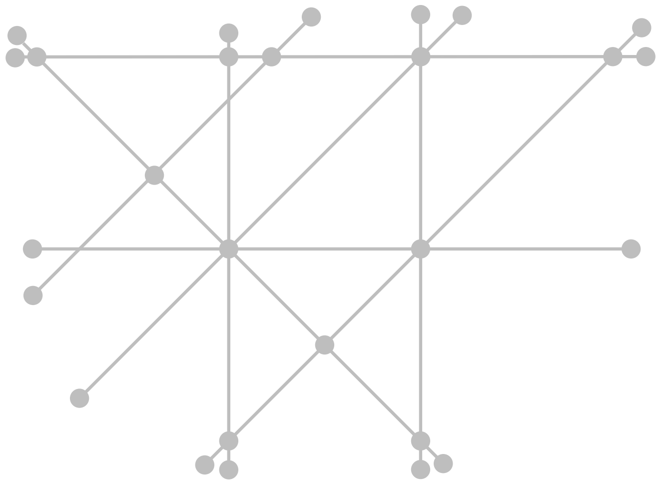


1. For each arrangement of k lines $\mathcal{O}(2^{k^2})$ arrangements

   Construct all plane graphs on $\leq \binom{k}{2} + 2k$ vertices with $2k$ leaves,
   and all coverings with k edge-disjoint paths.
   Use Renegar to check wether they are stretchable.



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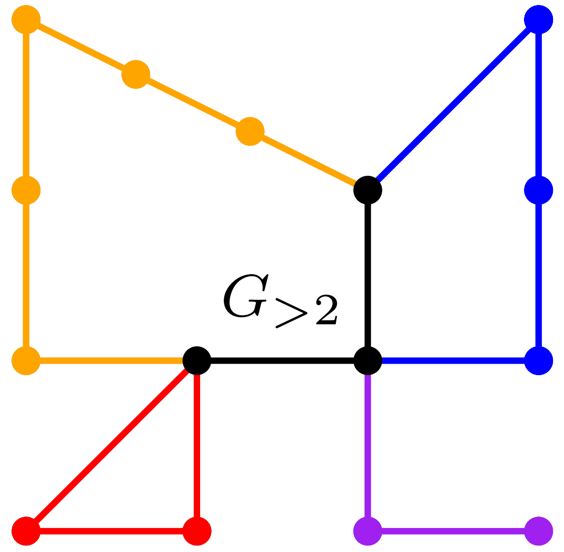

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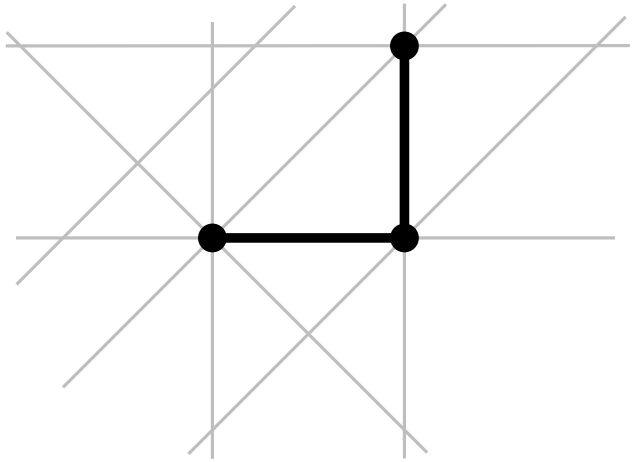
   2. For each placement of $V_{>2}$ on crossings of the lines

      If this yields a planar drawing of $G_{>2}$ with edges on the lines



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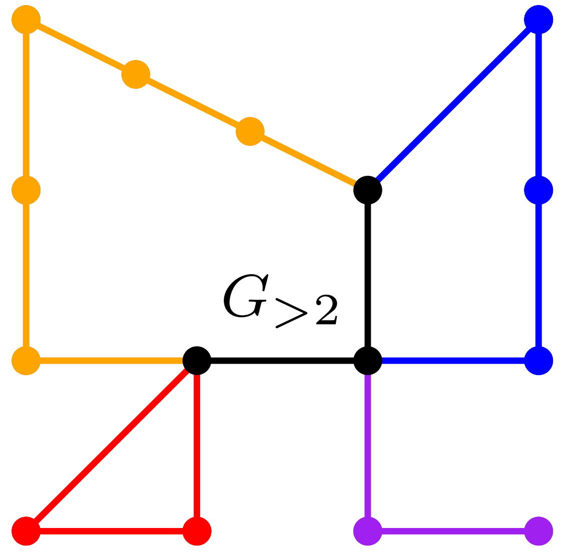

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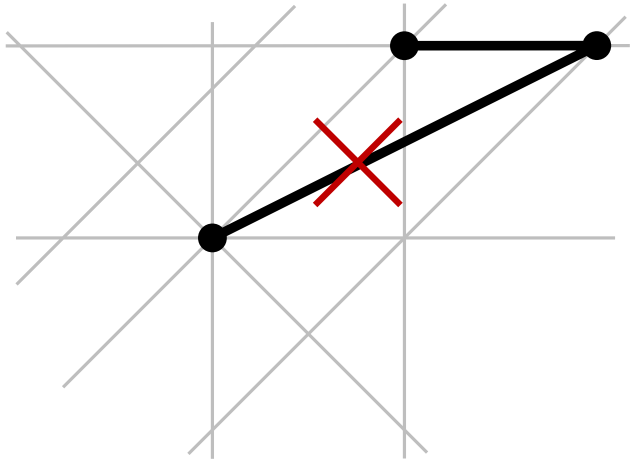

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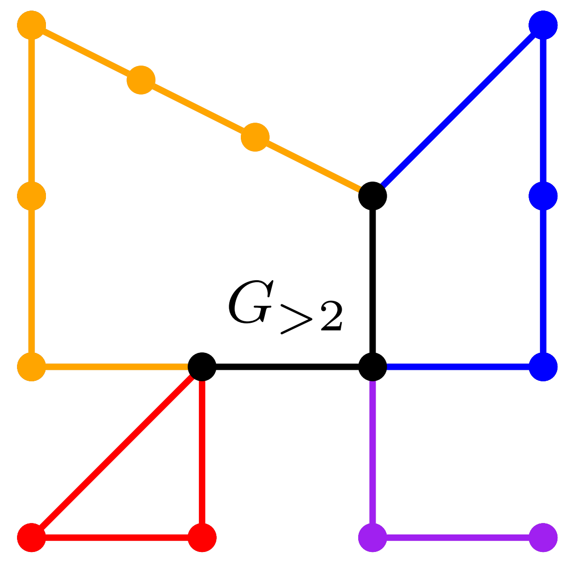

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   Construct all plane graphs on $\leq \binom{k}{2} + 2k$ vertices with $2k$ leaves,
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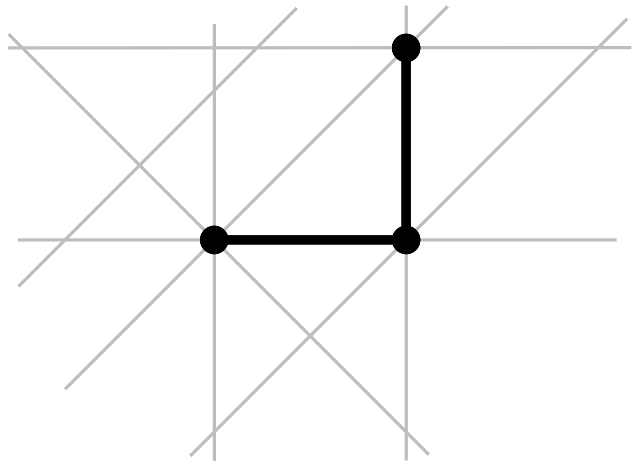
2. For each placement of $V_{>2}$ on crossings of the lines $\binom{k}{2}!$ many

   If this yields a planar drawing of $G_{>2}$ with edges on the lines



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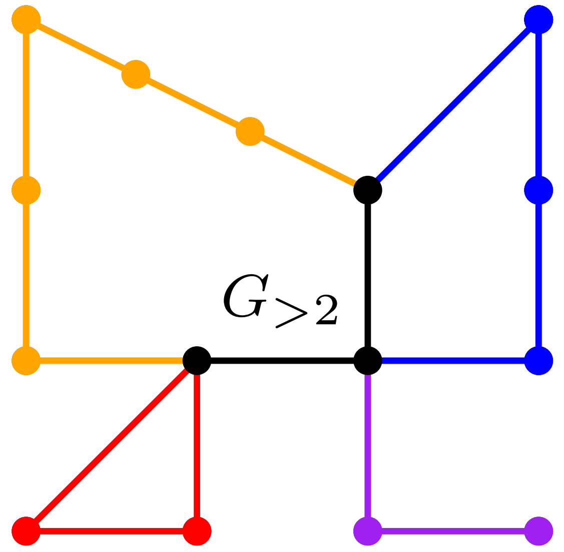

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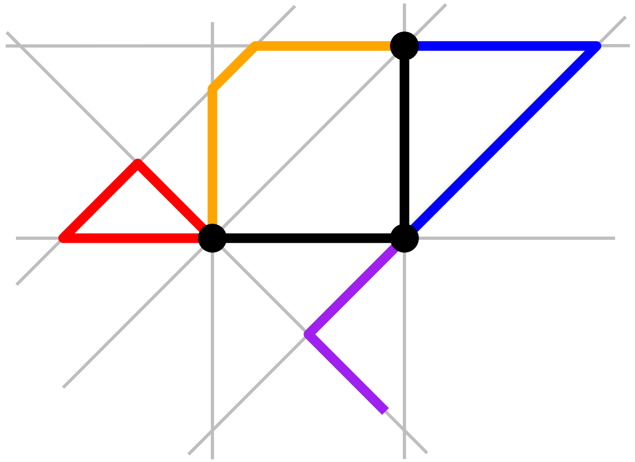
      If this yields a planar drawing of $G_{>2}$ with edges on the lines

      3. for each routing of the degree-2-paths that yield a planar drawing of G with $\leq k$ segments



List-Incidence Segment Number is in FPT wrt. Segment Number

Input: planar graph $G = (V, E)$, integer k, lists $L(e) \subseteq [k]$, $e \in E$

Split G into $|V_{>2}| \leq \binom{k}{2}$

- graph $G_{>2} = (V_{>2}, E_{>2})$ induced by vertices of degree > 2
- light paths of degree 2. at most $2k \cdot \binom{k}{2}$ many

1. For each arrangement of k lines $\mathcal{O}(2^{k^2})$ arrangements

   Construct all plane graphs on $\leq \binom{k}{2} + 2k$ vertices with $2k$ leaves,
   and all coverings with k edge-disjoint paths.
   Use Renegar to check wether they are stretchable.

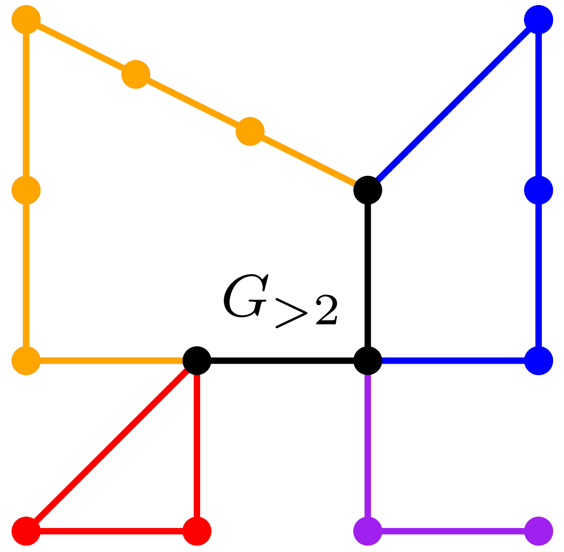
   2. For each placement of $V_{>2}$ on crossings of the lines $\binom{k}{2}!$ many

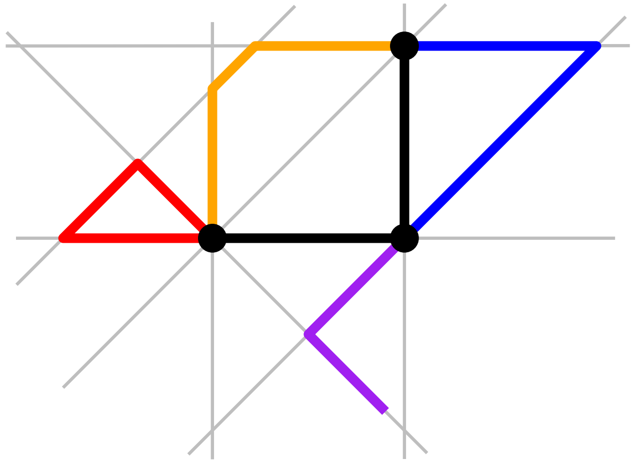
      If this yields a planar drawing of $G_{>2}$ with edges on the lines

      3. for each routing of the degree-2-paths that yield a planar drawing of G with $\leq k$ segments

$\left((2k(k-1))^k\right)^{k^3}$ many

- $2k(k-1)$: number of choices per path and crossing
- exponent k: number of segments per light path
- exponent k^3: number of light paths





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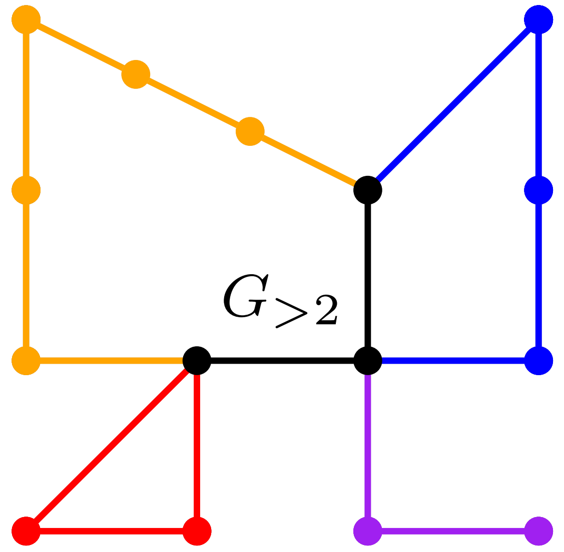

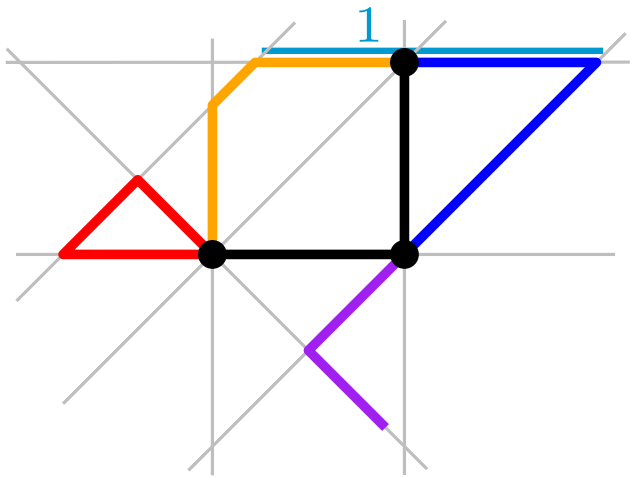

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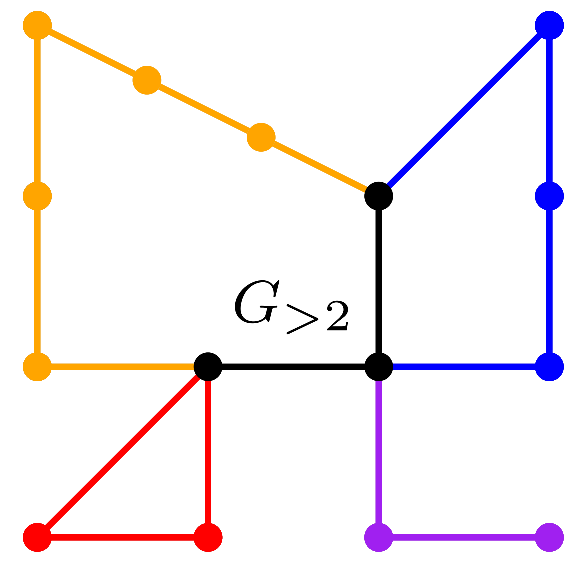

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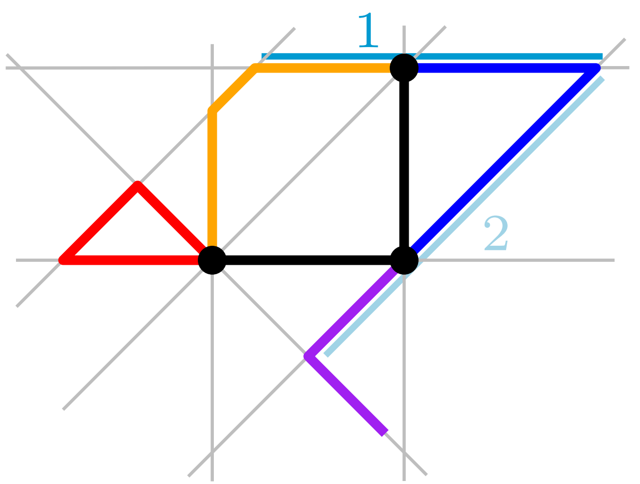

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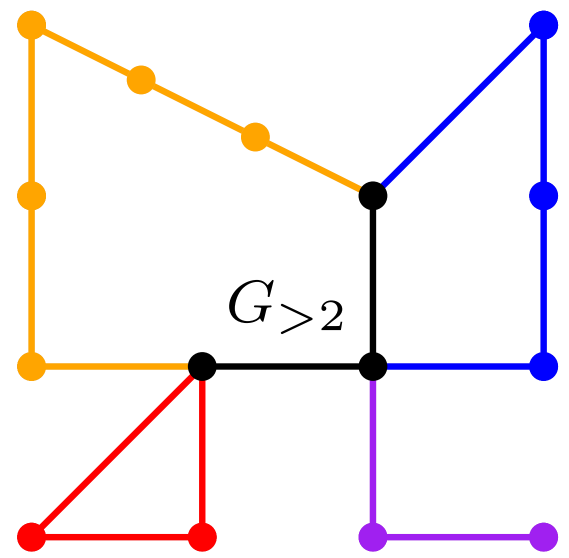

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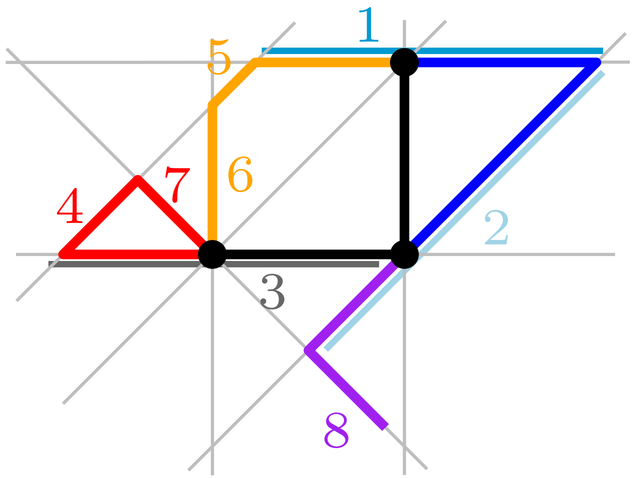

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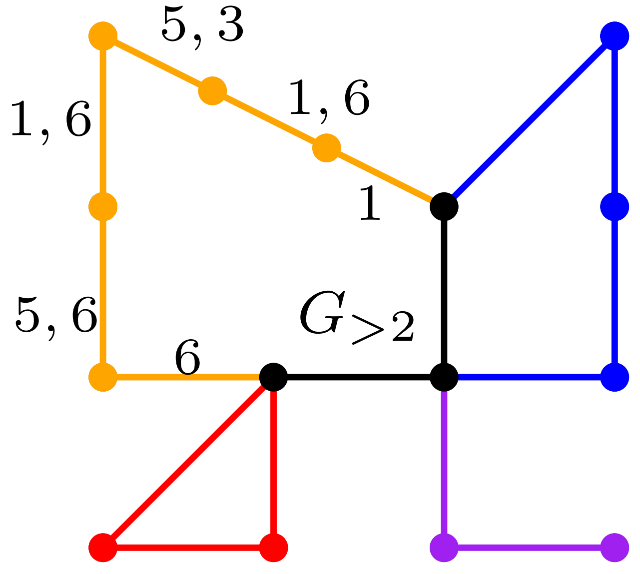

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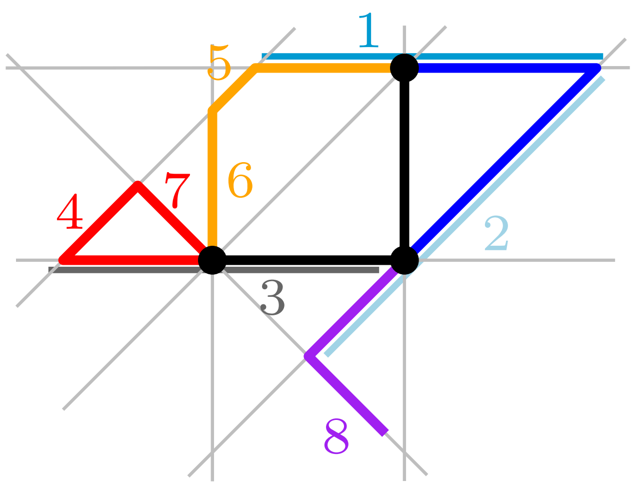

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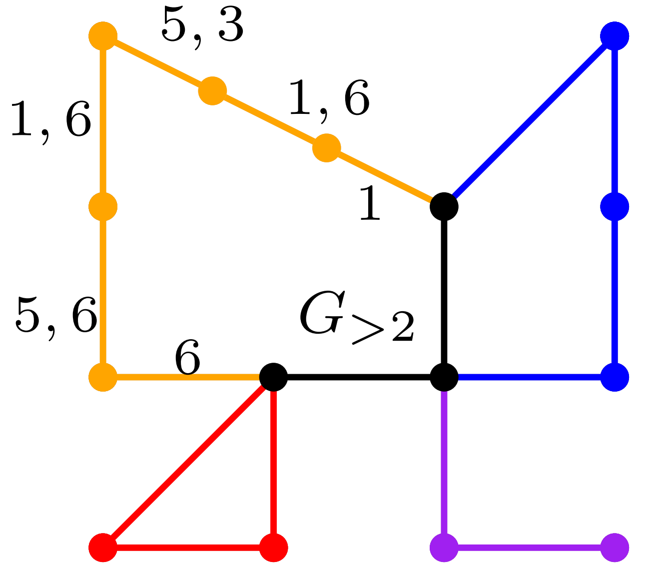

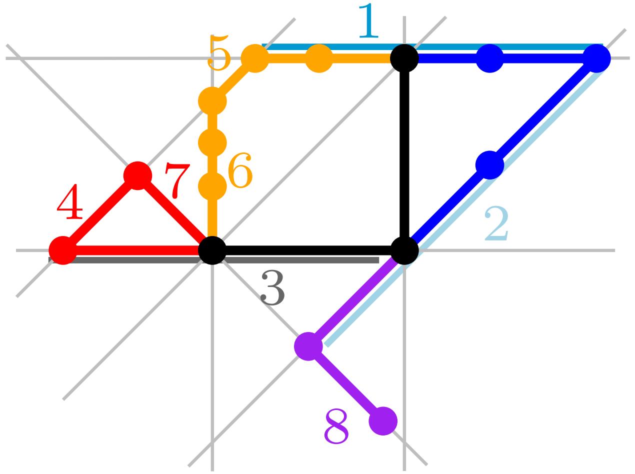

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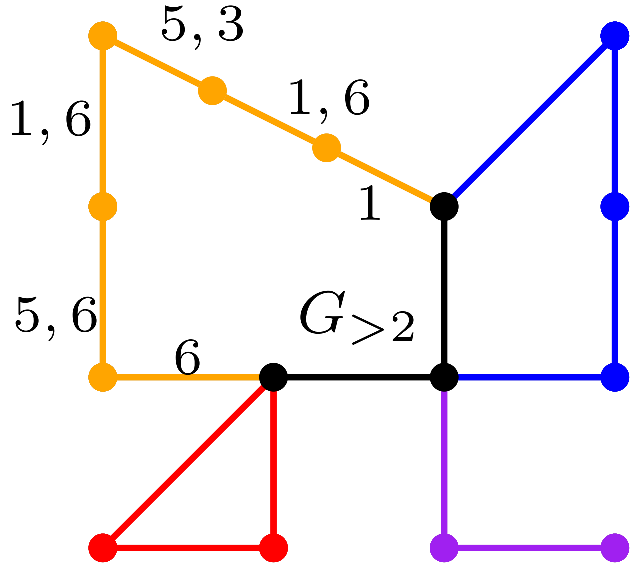
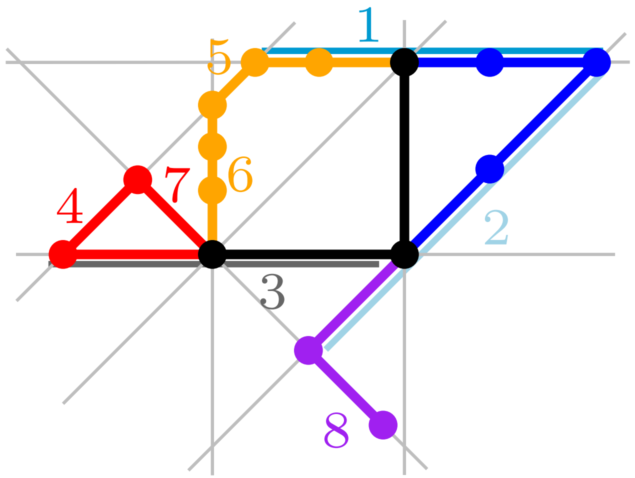
   $\mathcal{O}(|P| \cdot |S| \cdot \log k)$ per path $\rightsquigarrow \mathcal{O}(n \cdot k \cdot \log k)$ for all paths





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  - treewidth (complexity even for tree-width 2 unknown)
  - cluster deletion number
    minimum number of vertices that have to be removed s.t.
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Thank you