Parameterized and Approximation Algorithms for the Maximum Bimodal Subgraph Problem

GD 2023, Palermo

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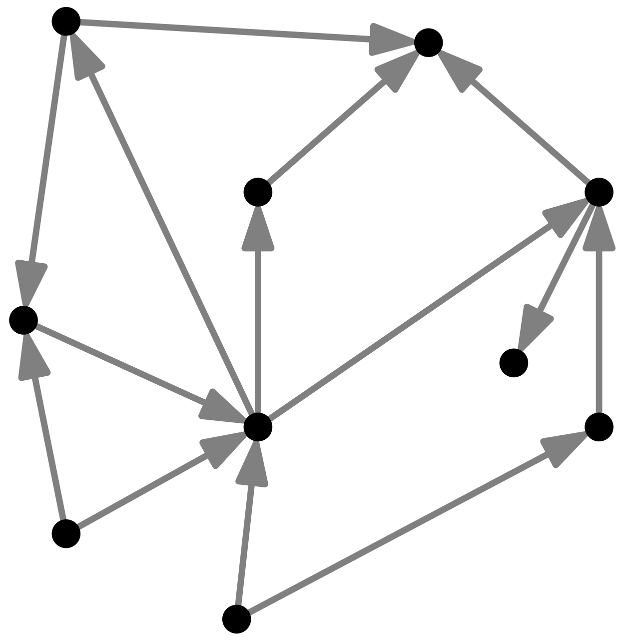
[4] University of Würzburg, Germany

Bimodality



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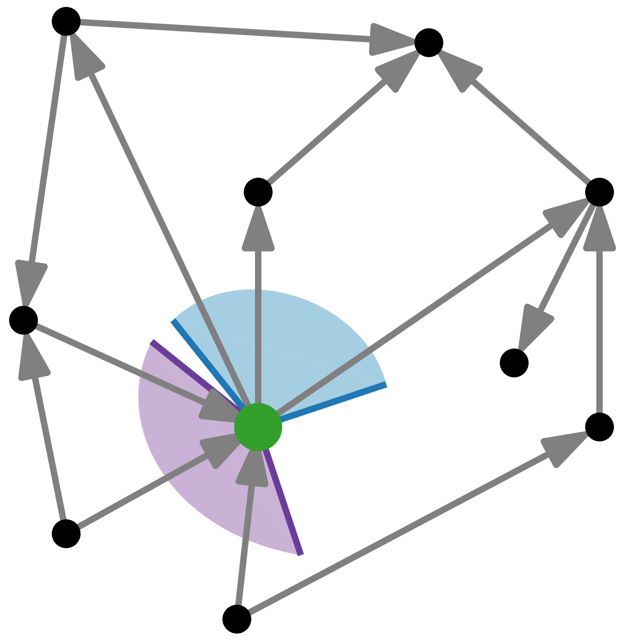

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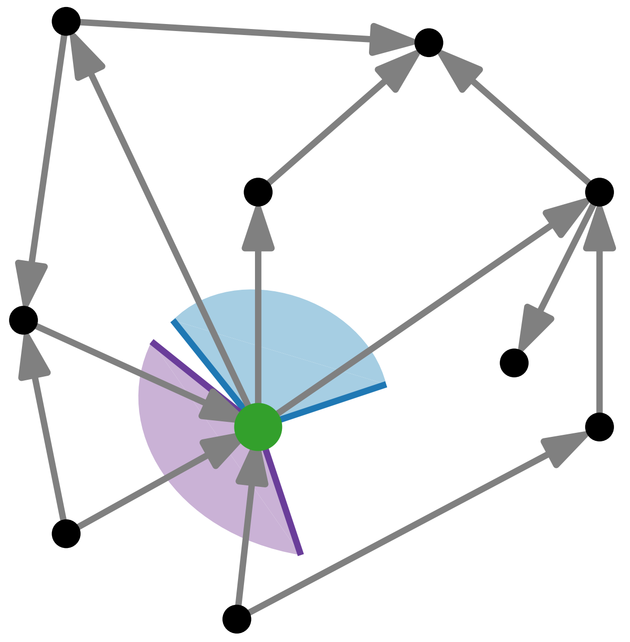

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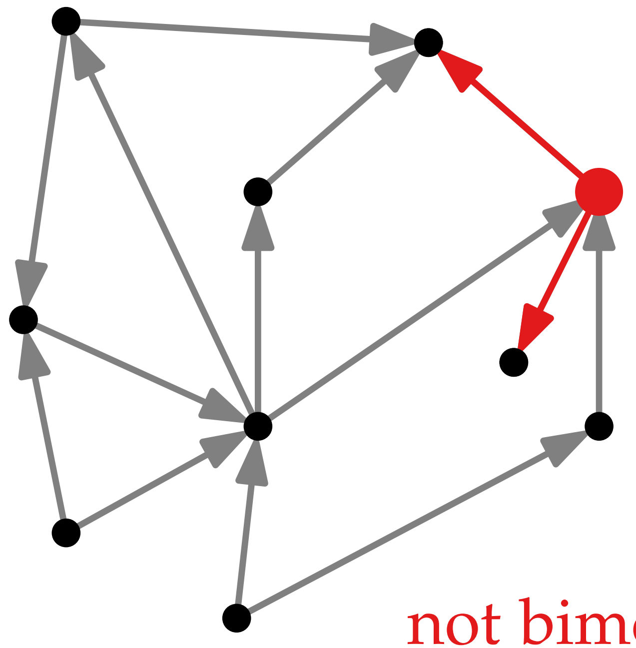

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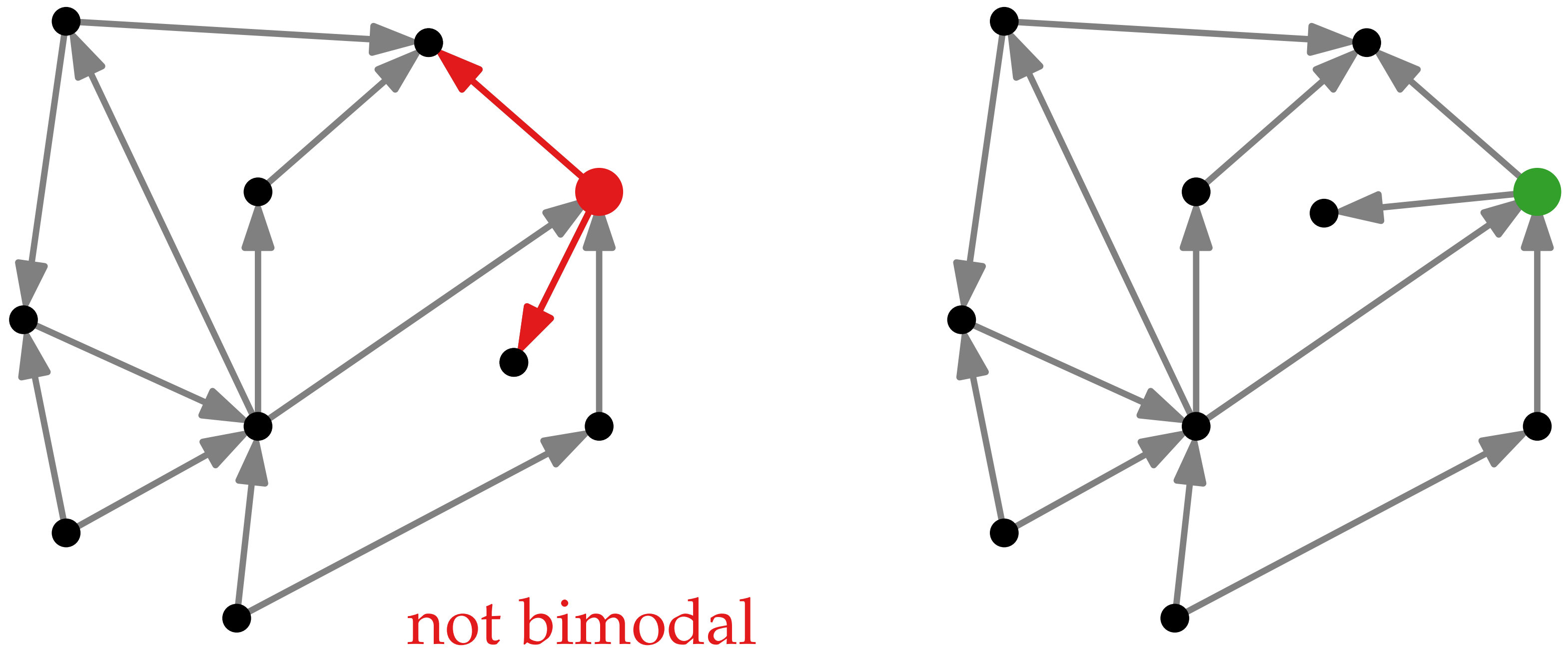

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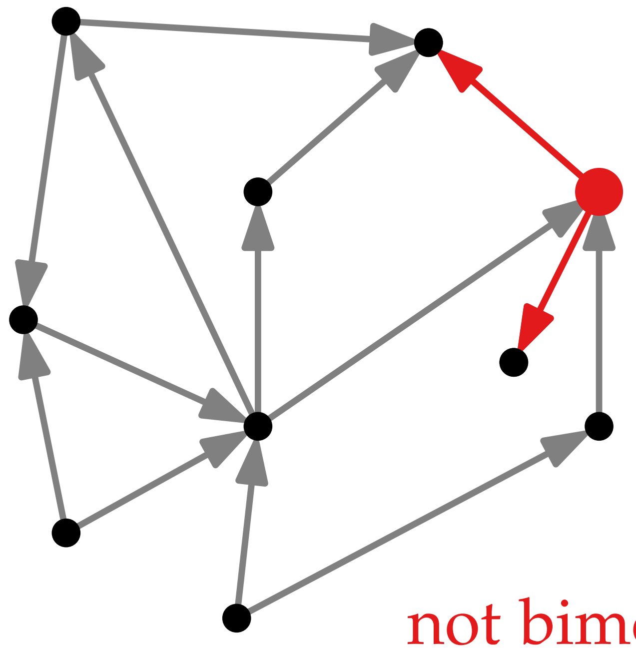

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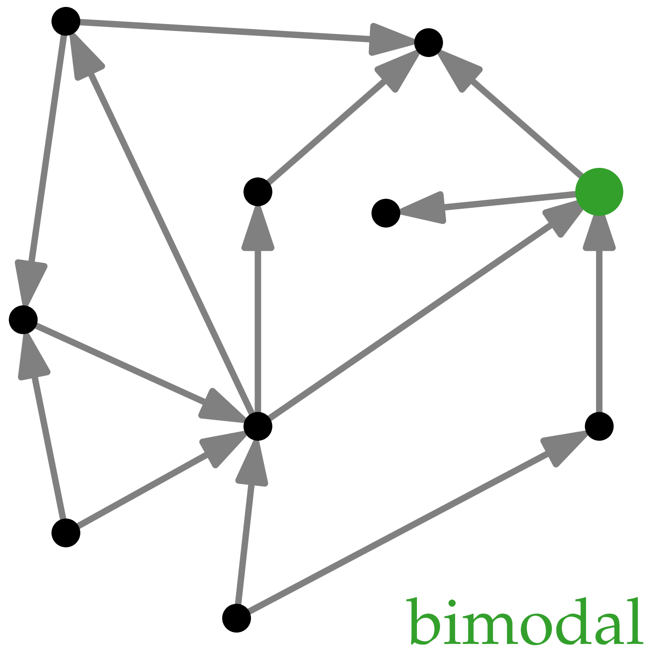

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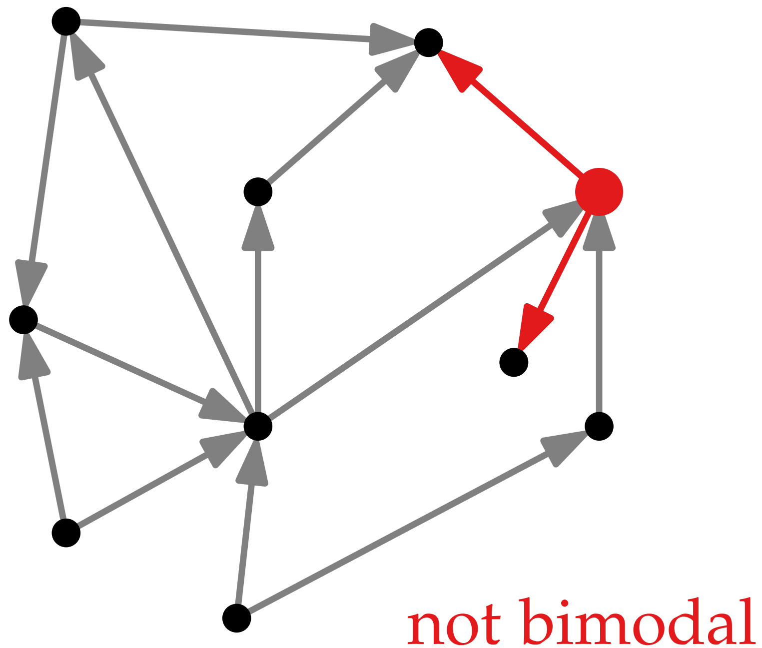

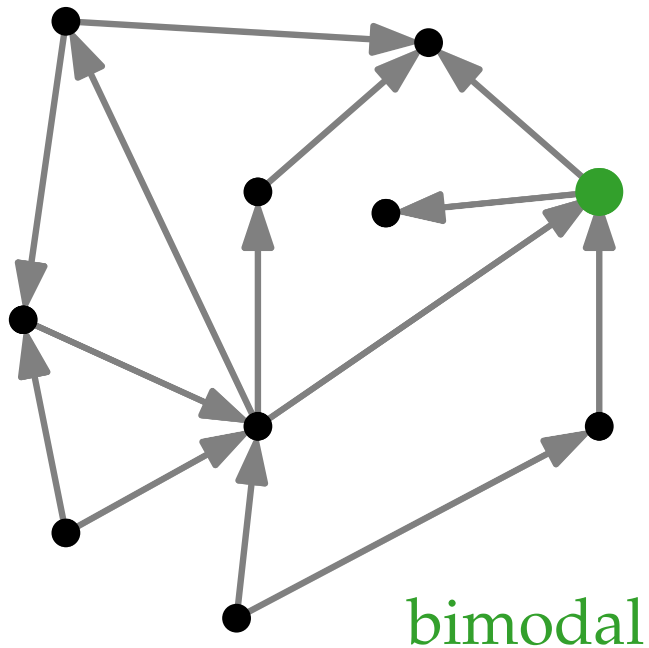

Embedding important!

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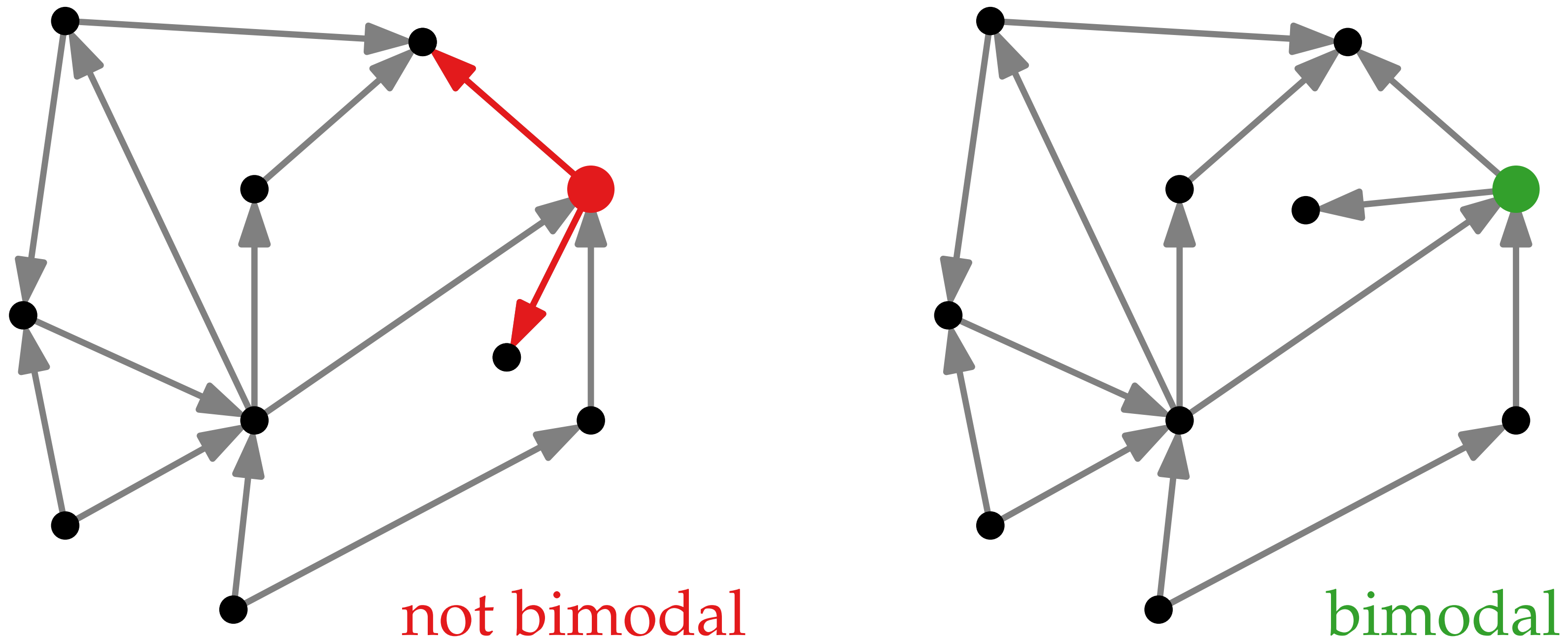

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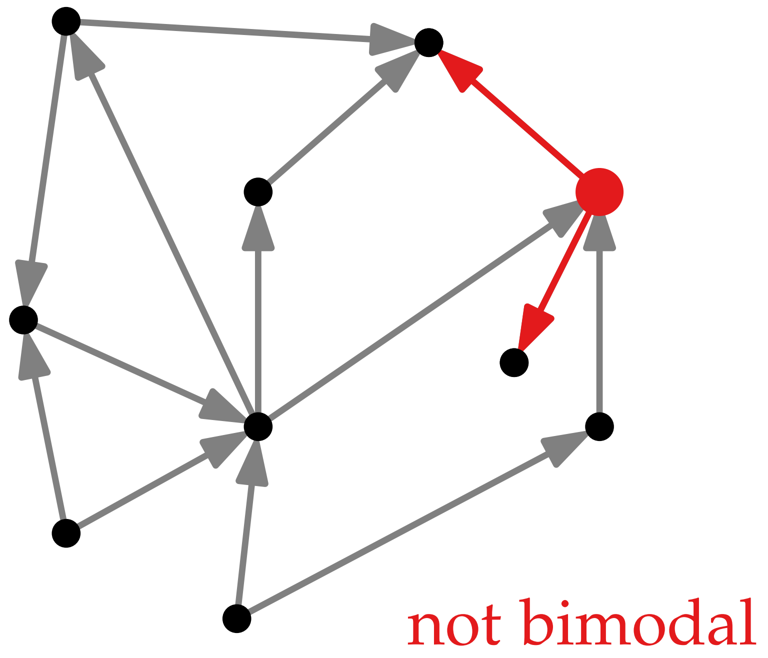

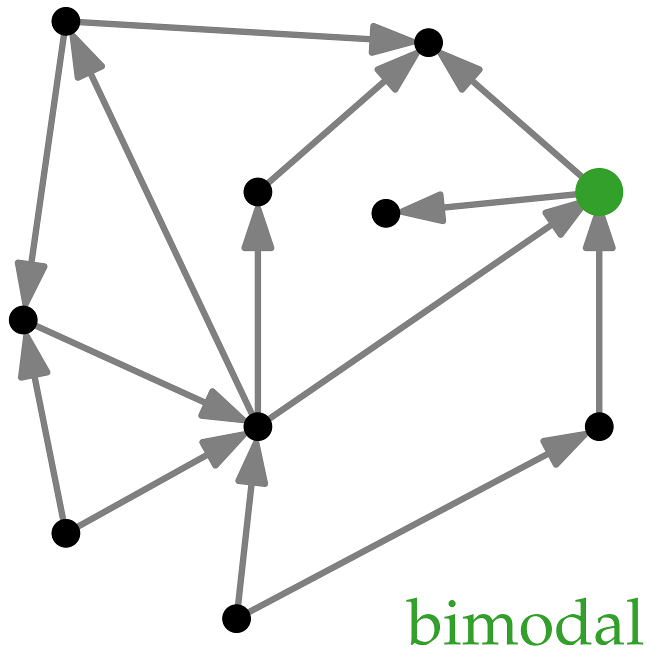

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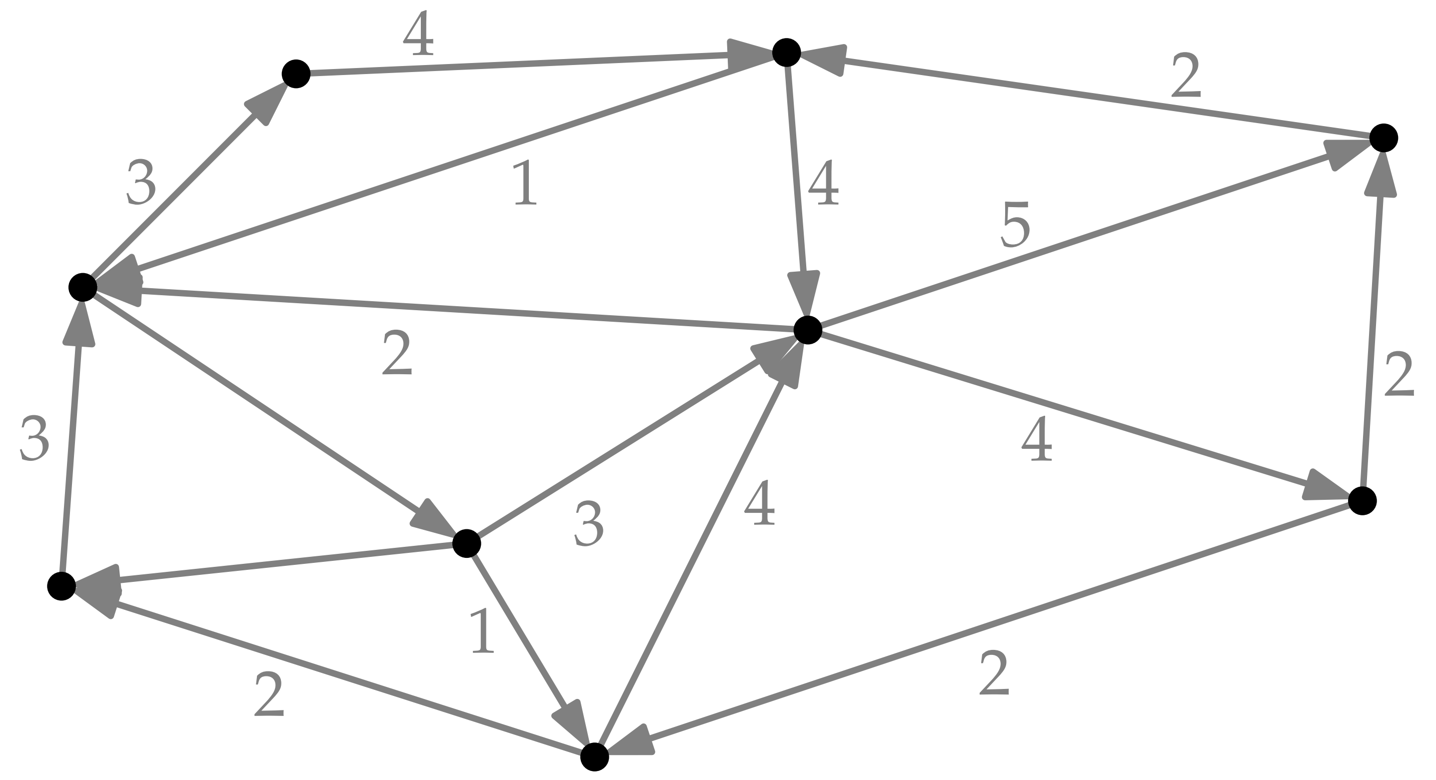

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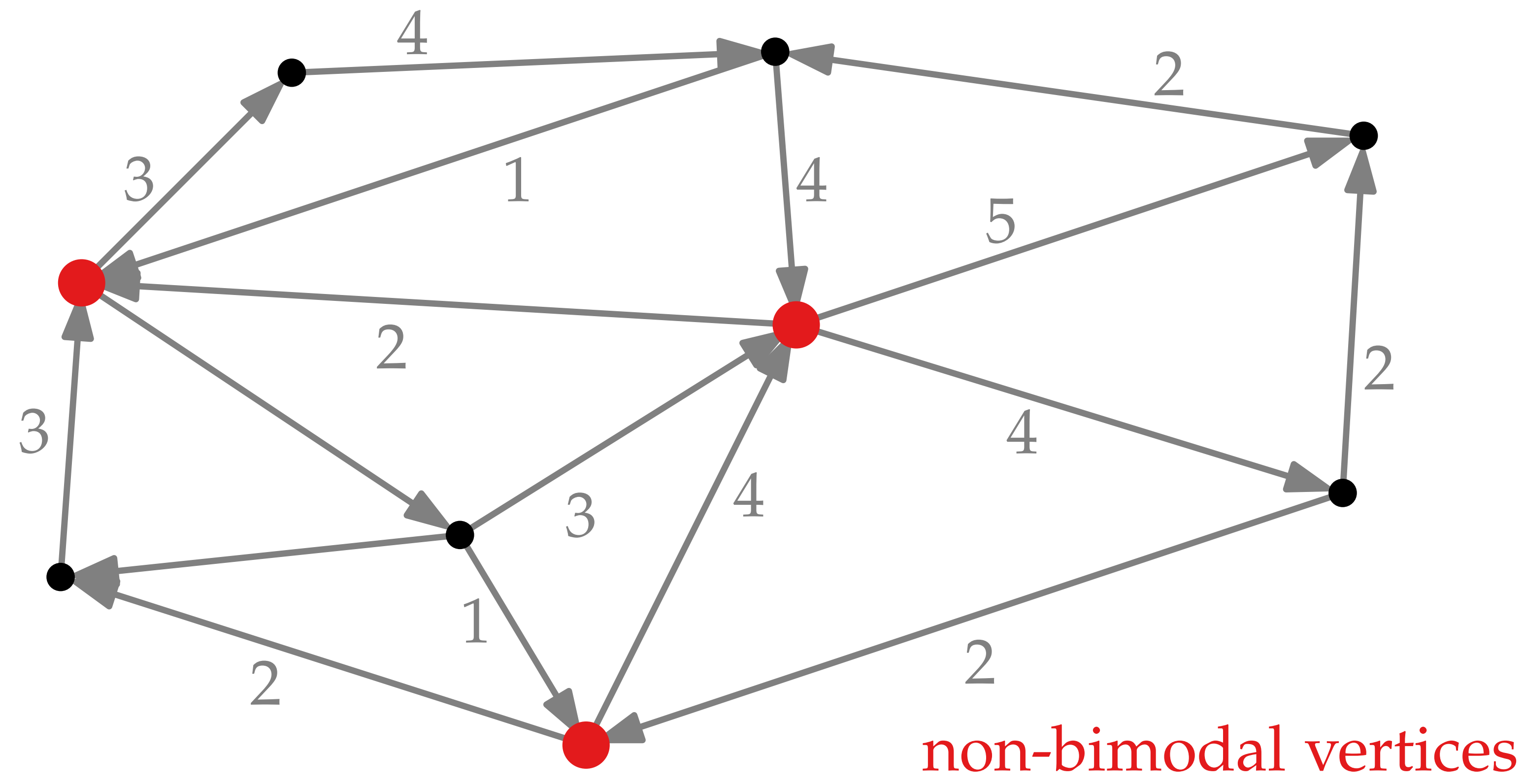

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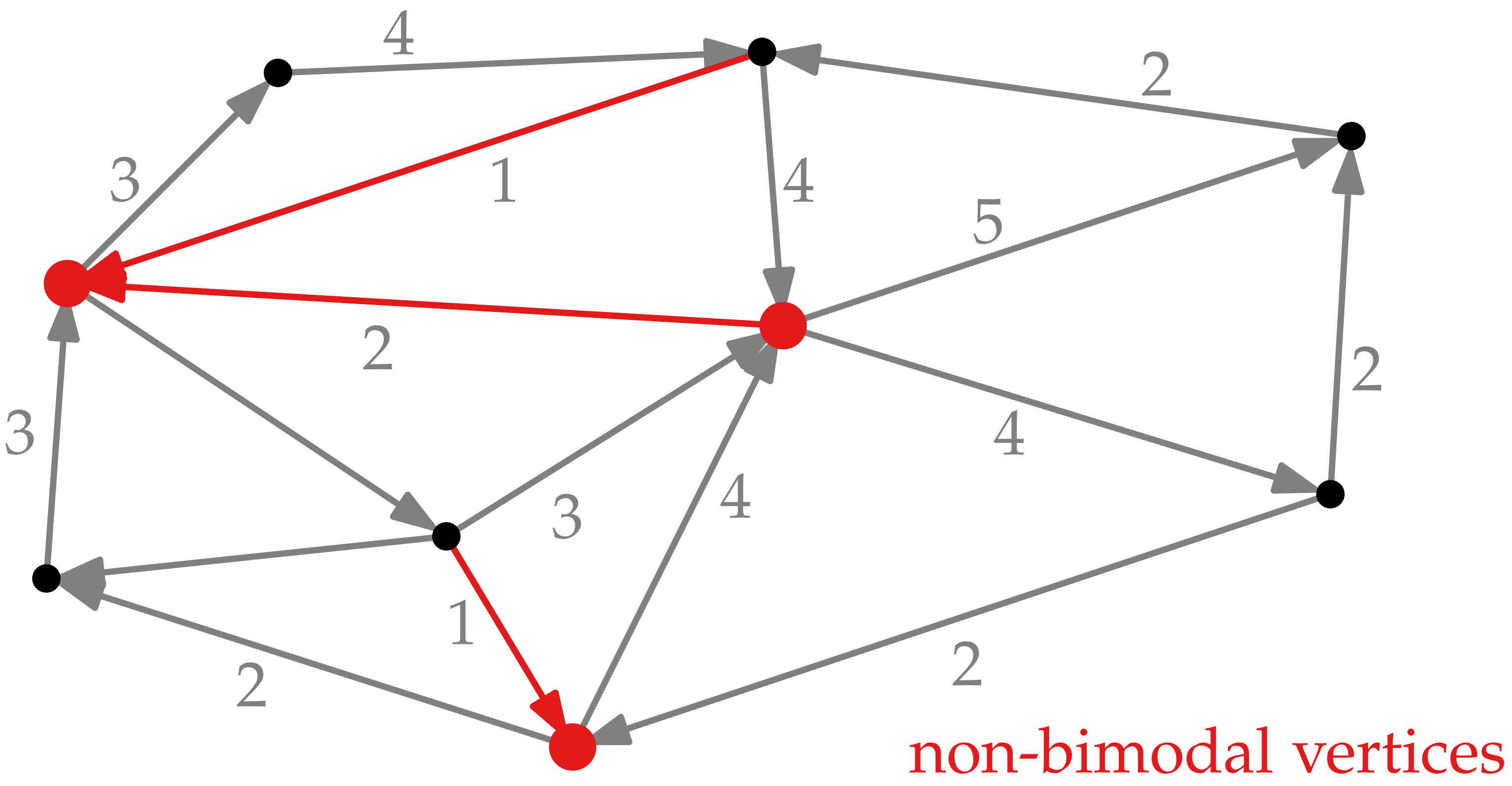

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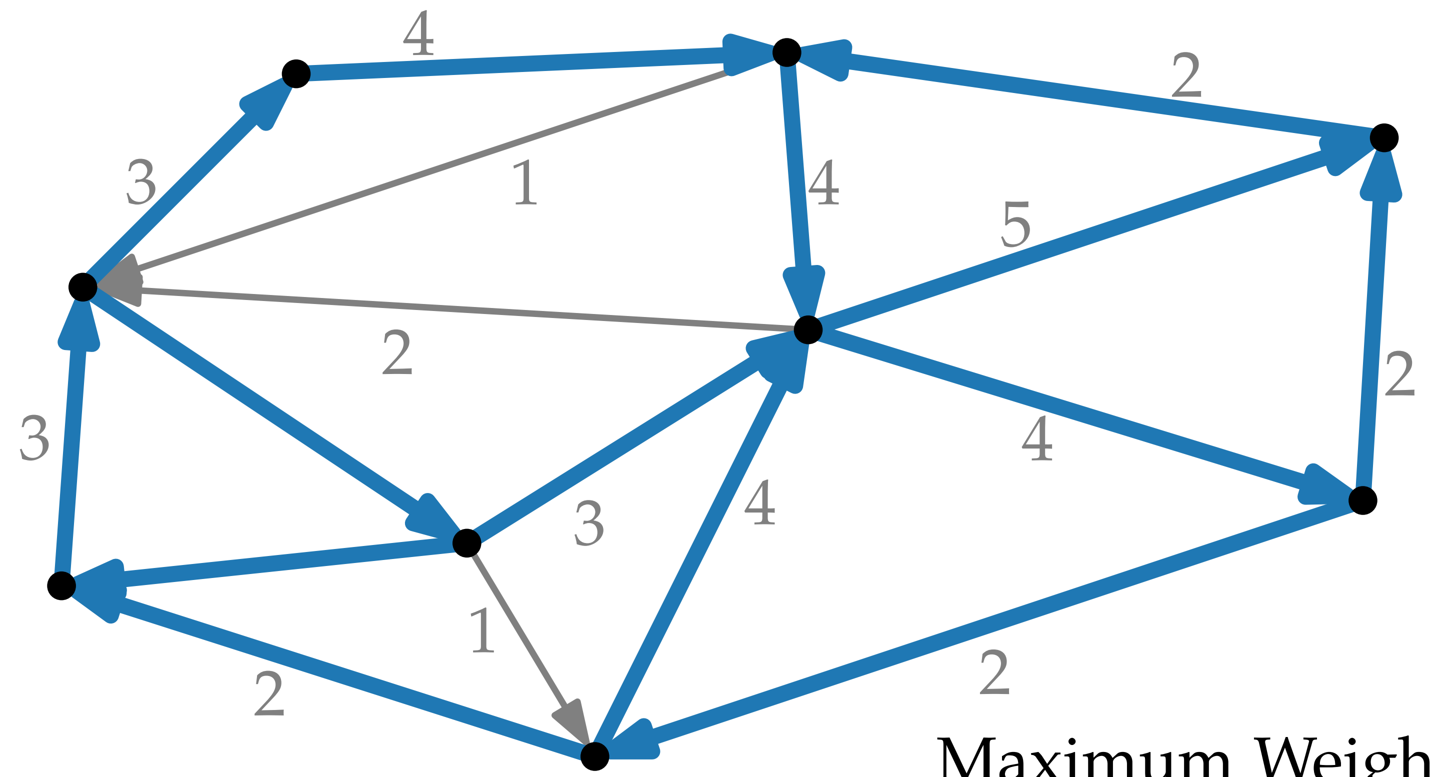

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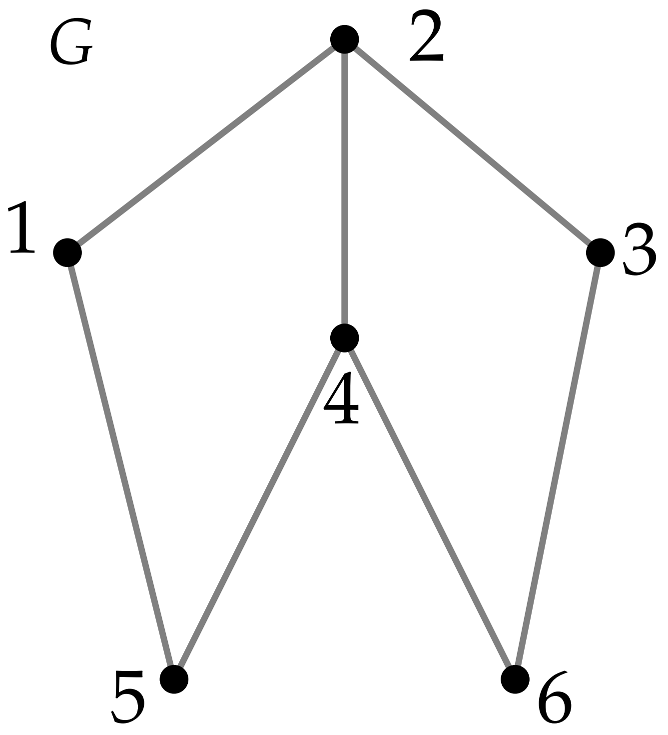

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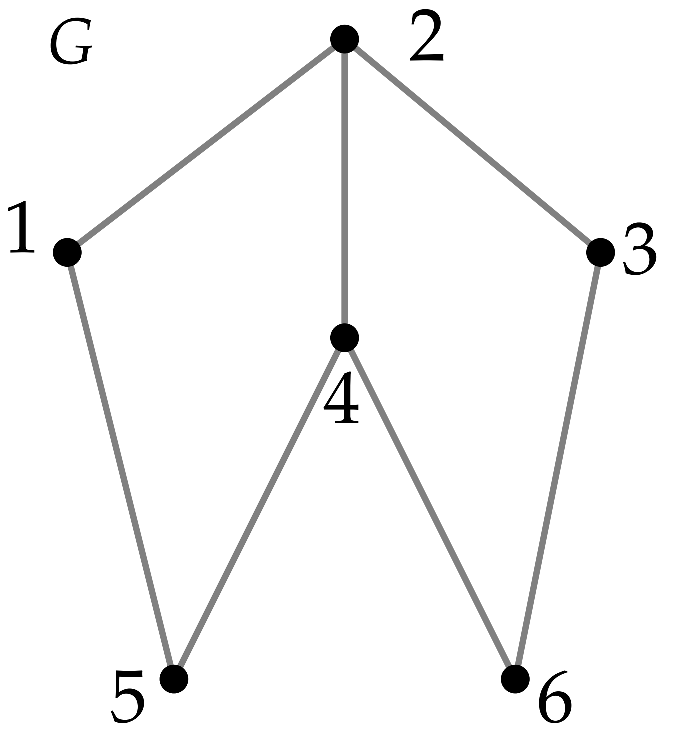

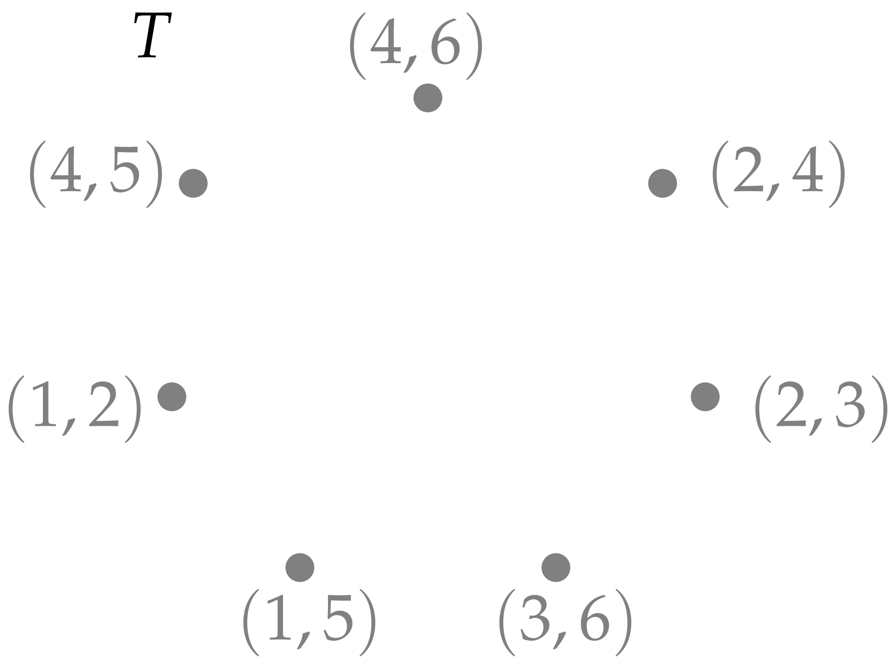

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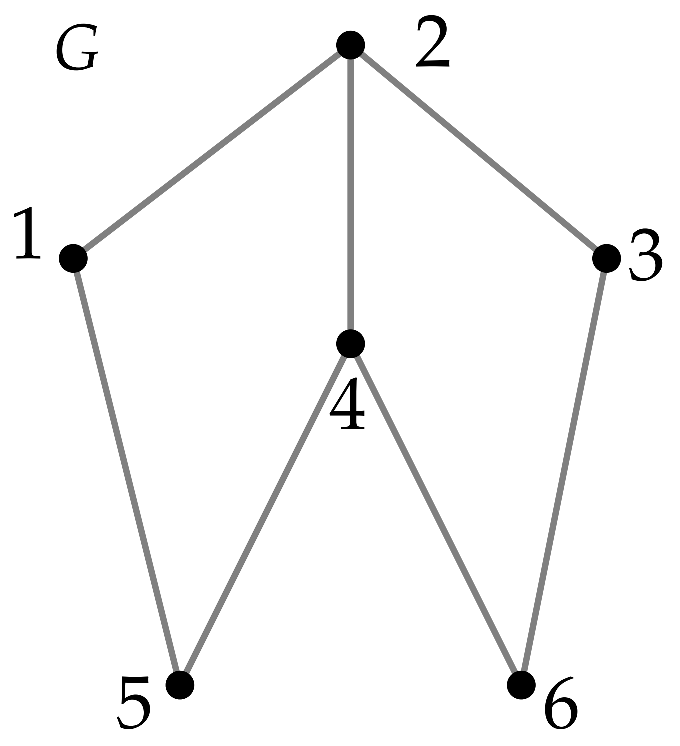

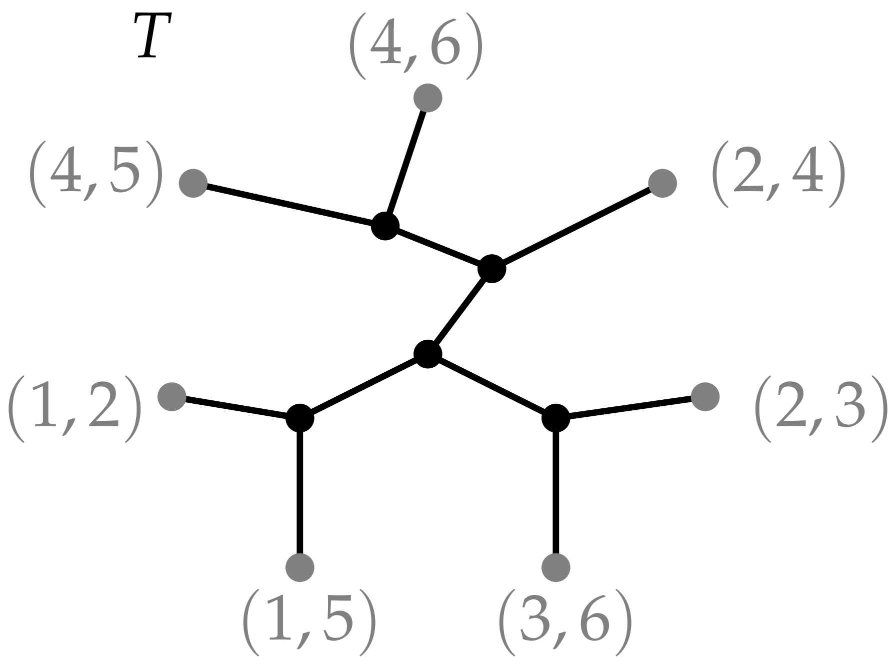

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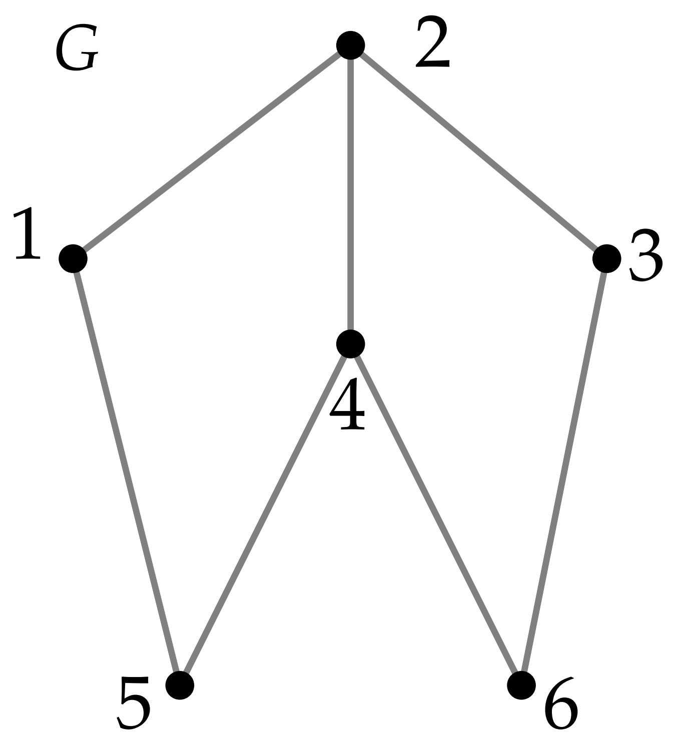

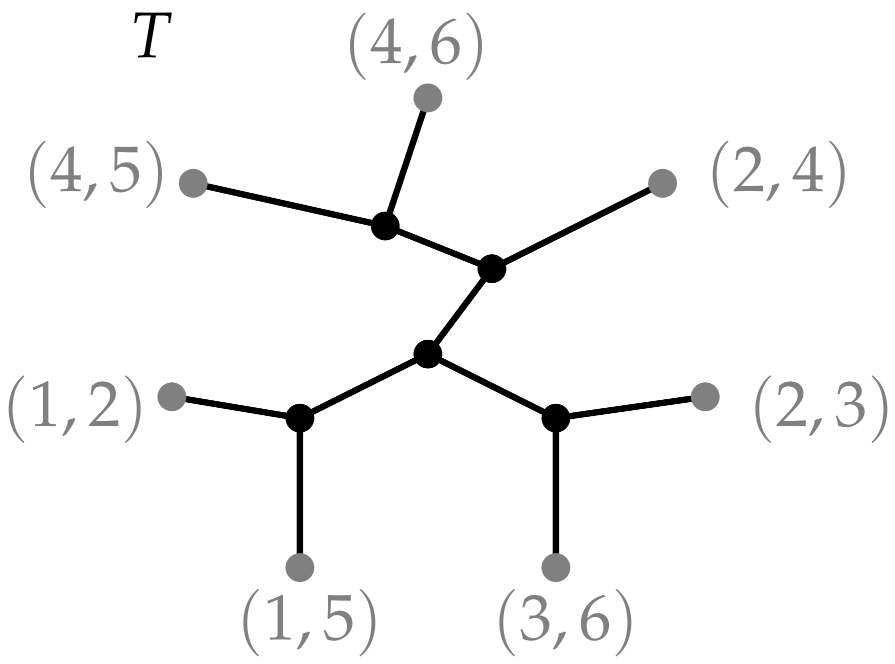

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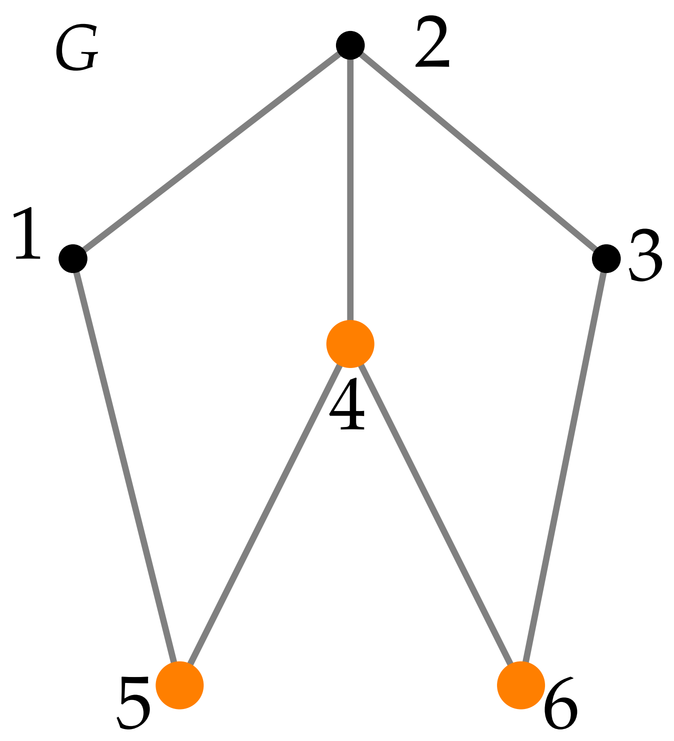

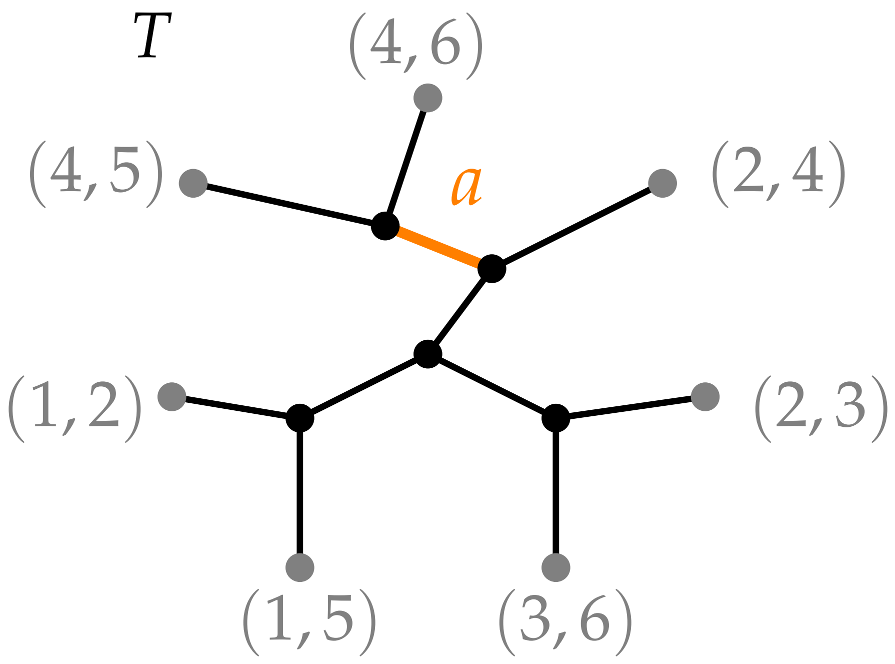

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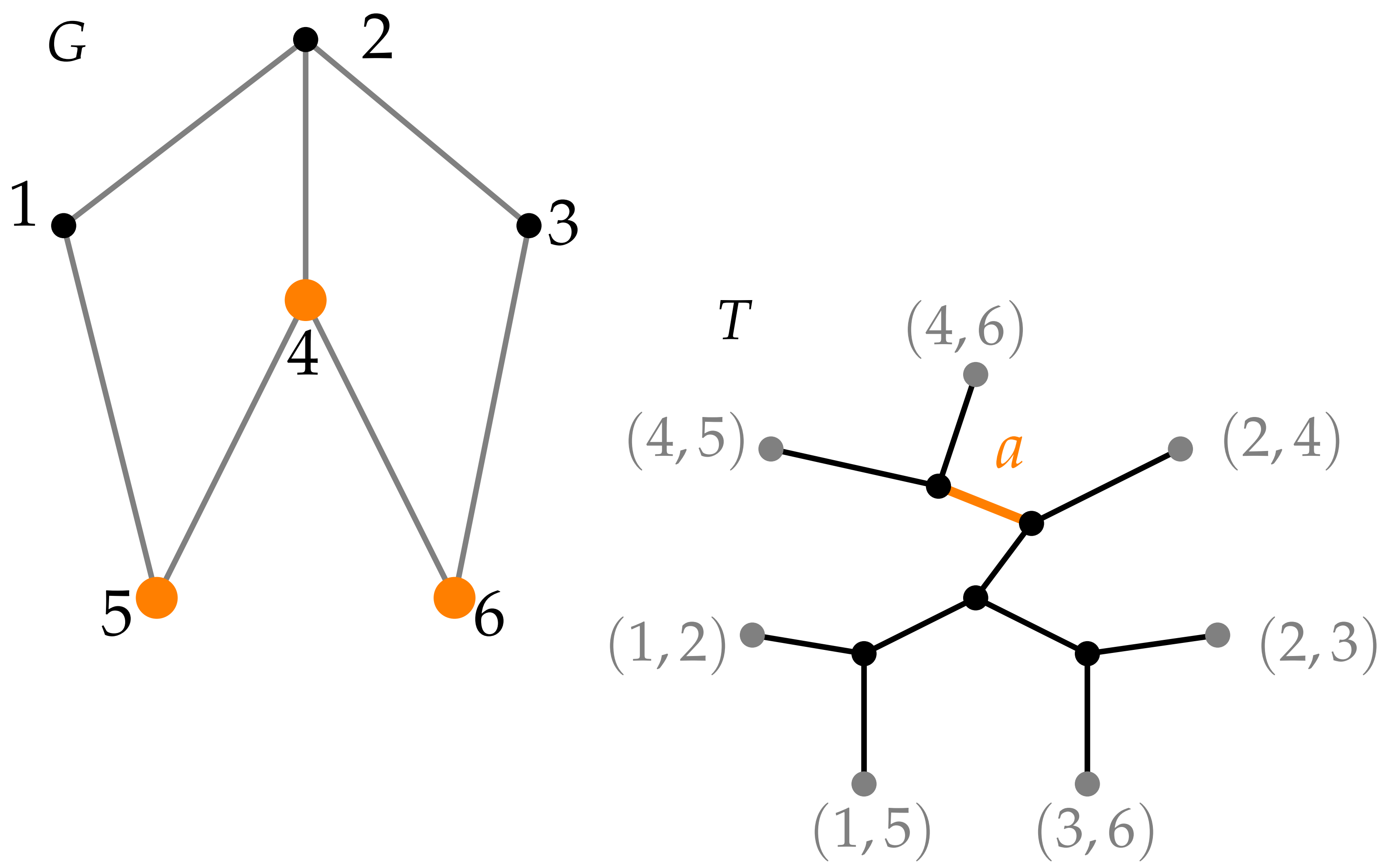

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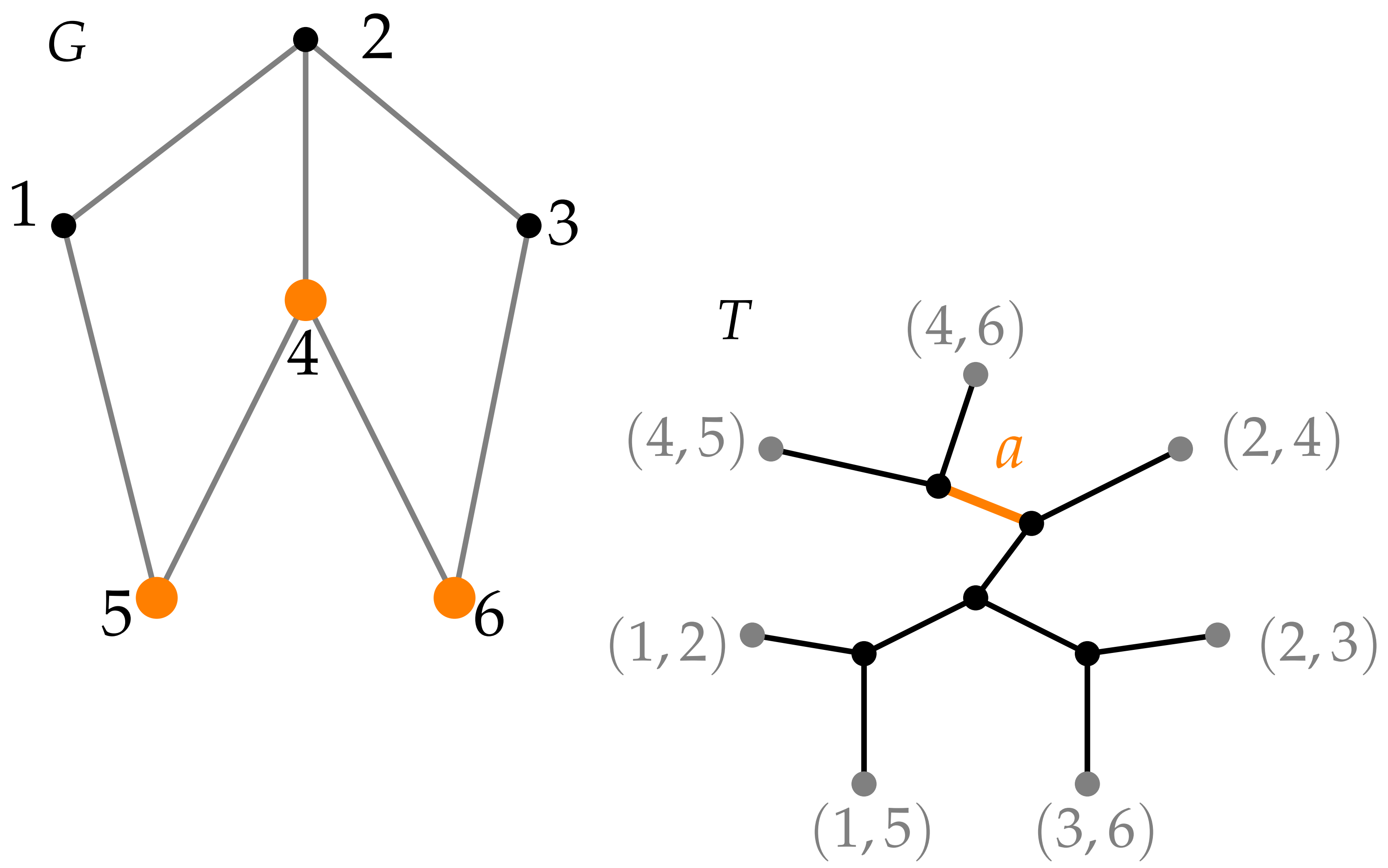

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The **branchwidth** $\mathrm{bw}(G)$ of G is the minimum width of all branch decompositions of G.

Sphere-Cut Decomposition

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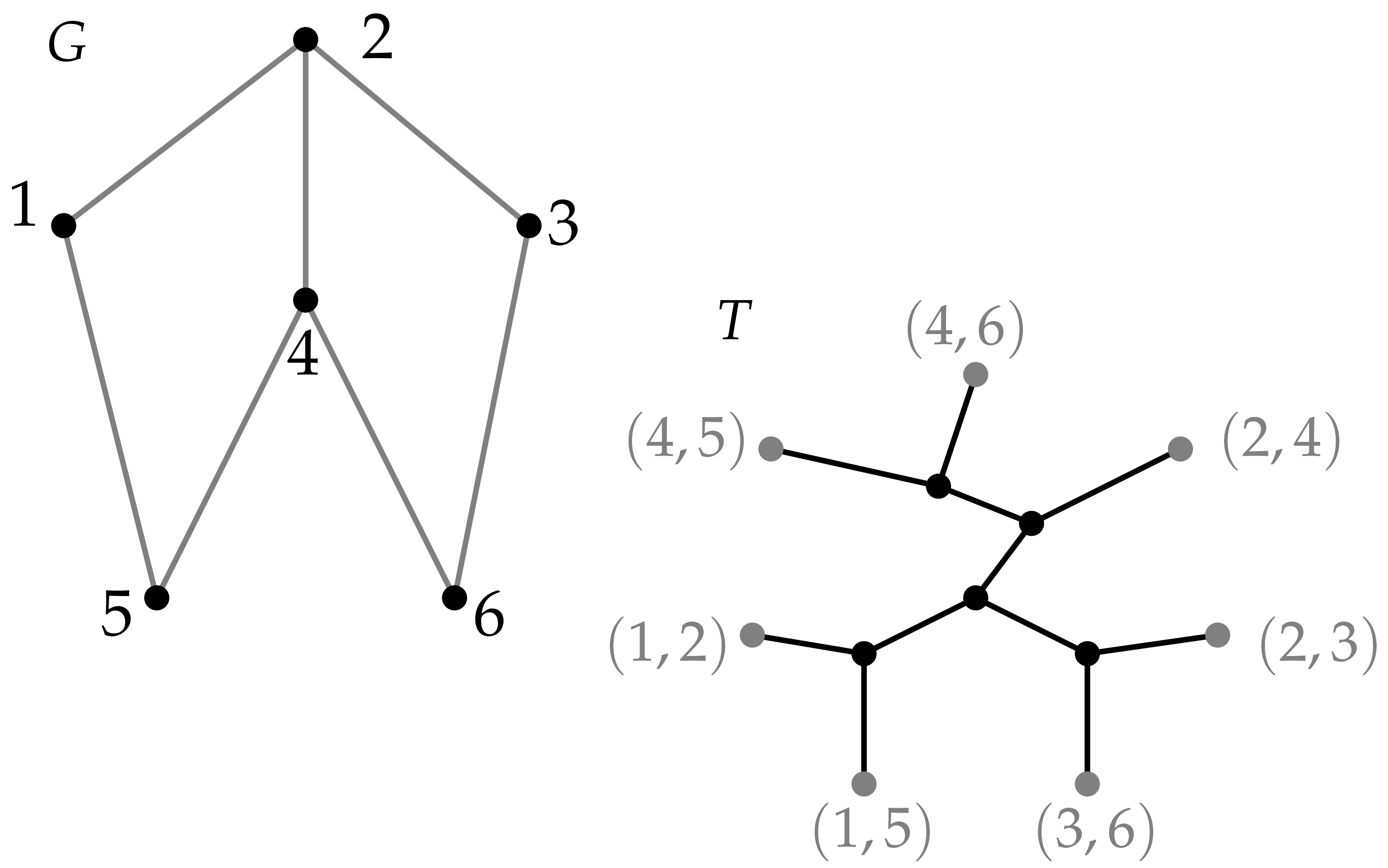

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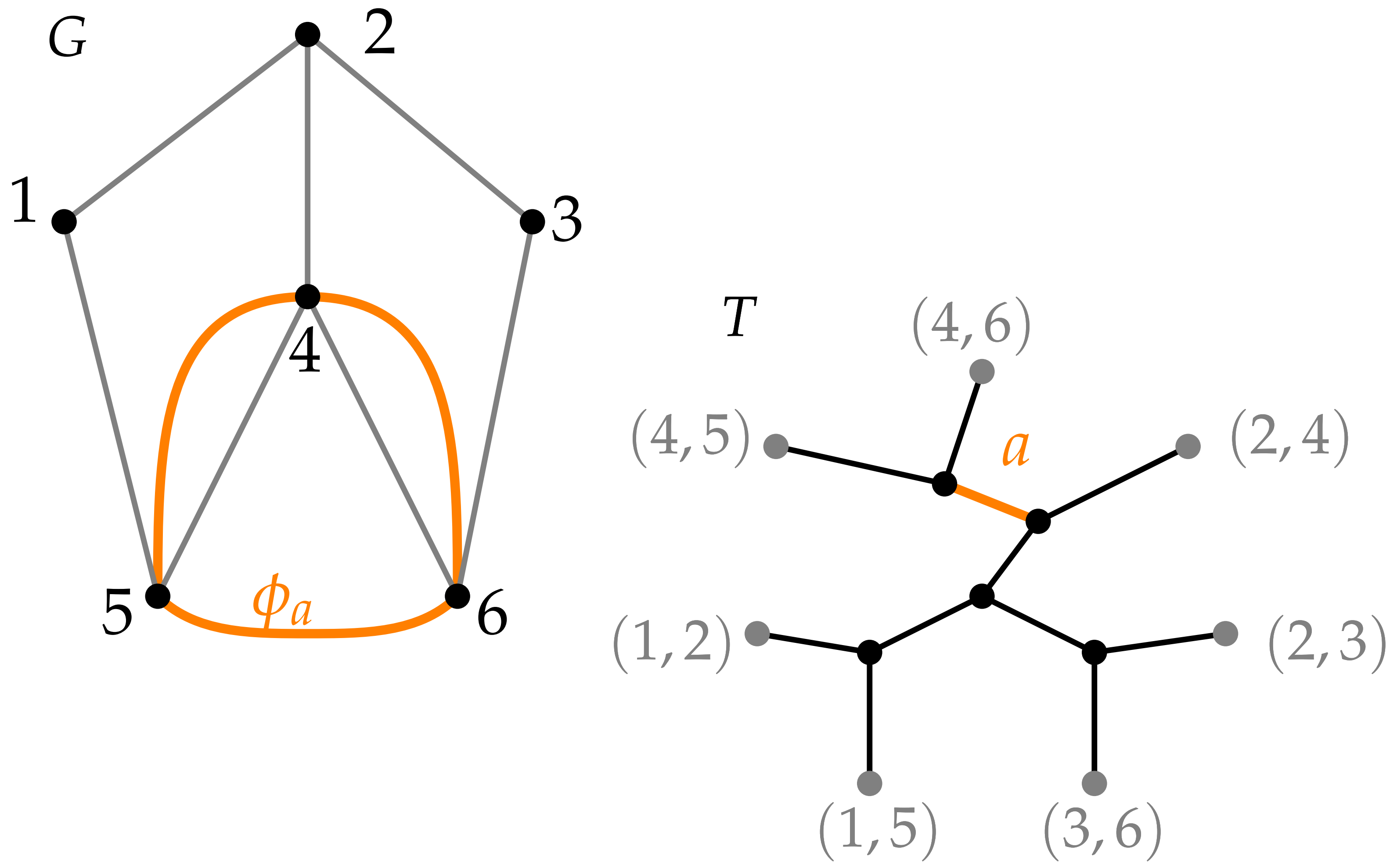

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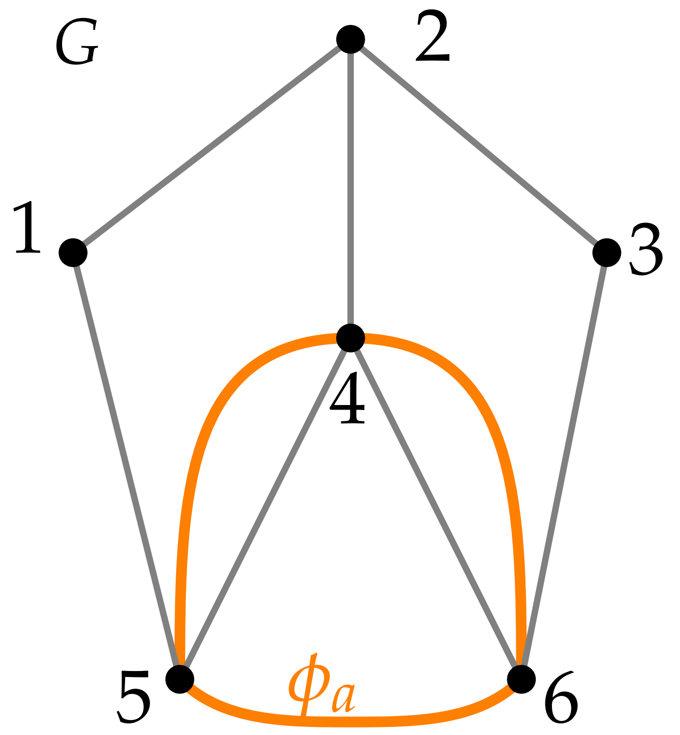

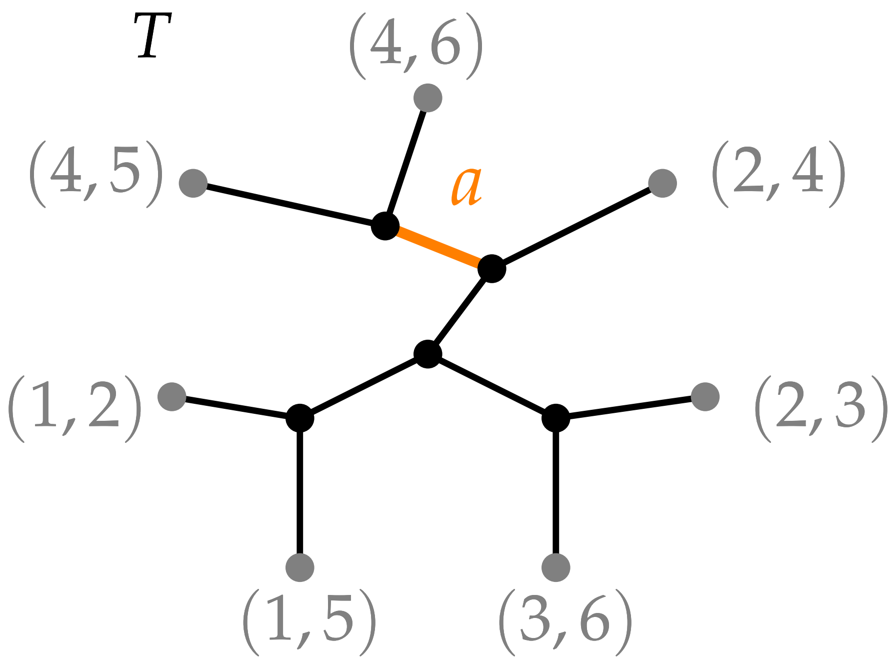

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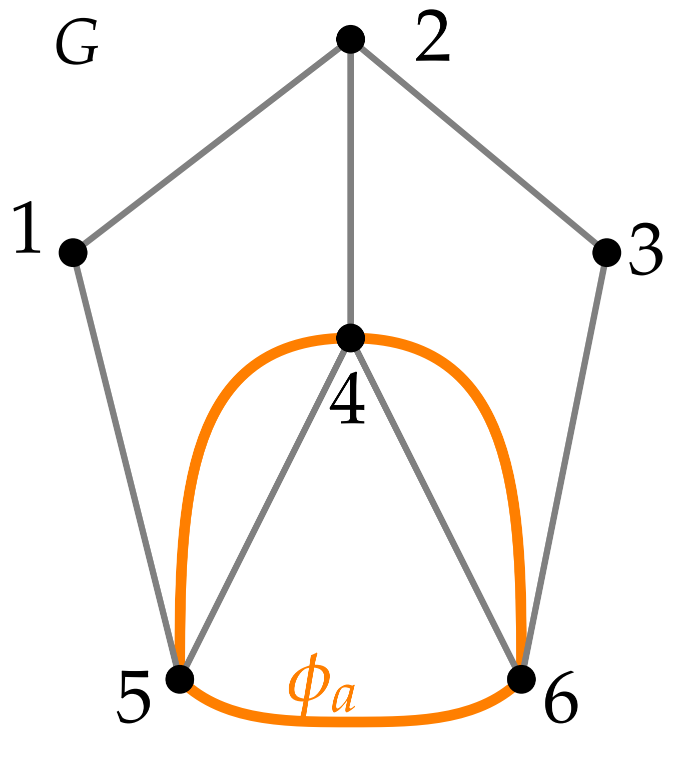

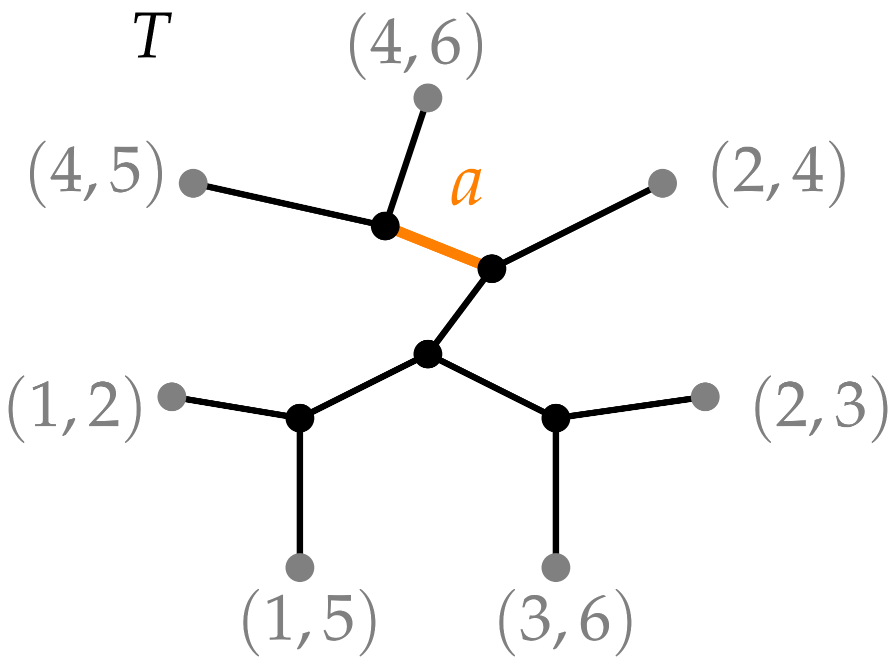

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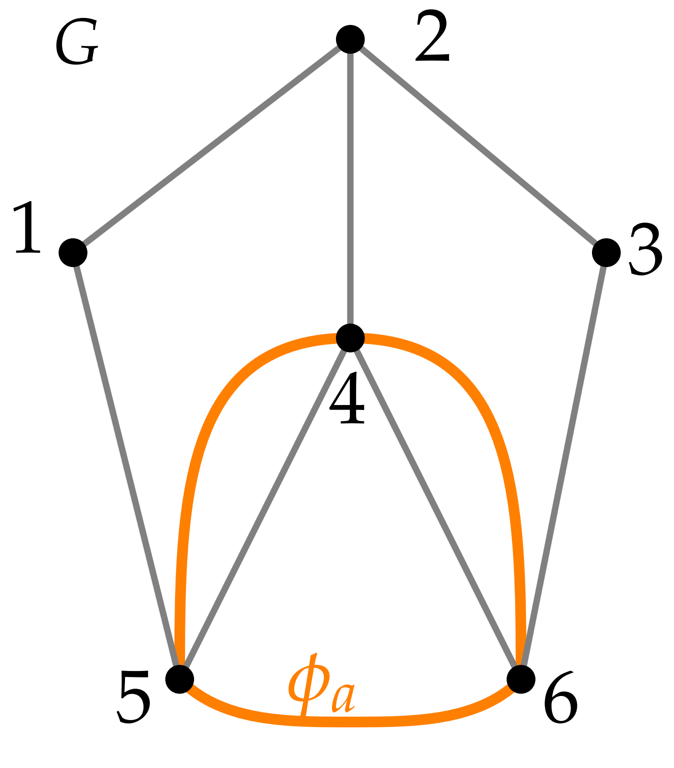

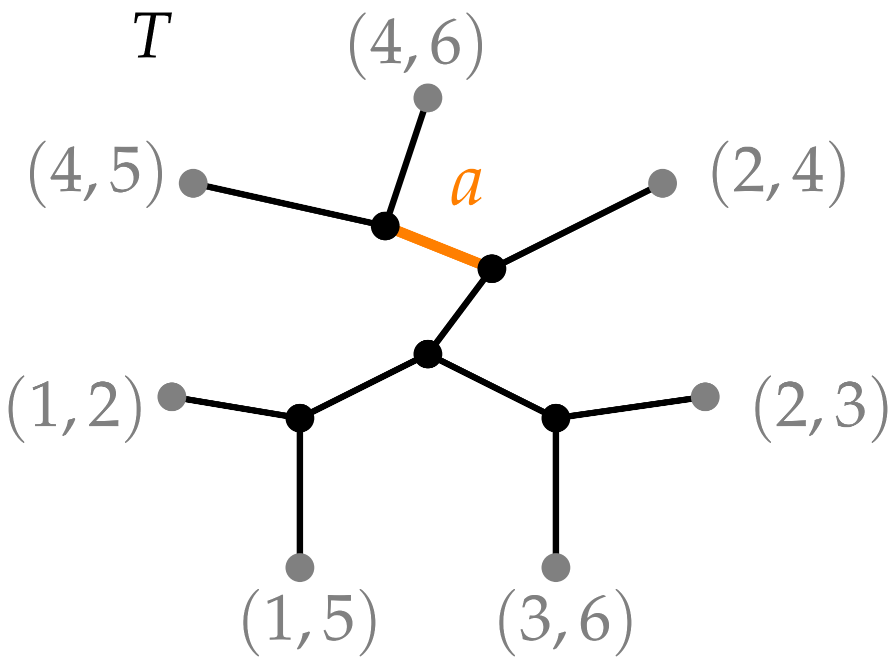

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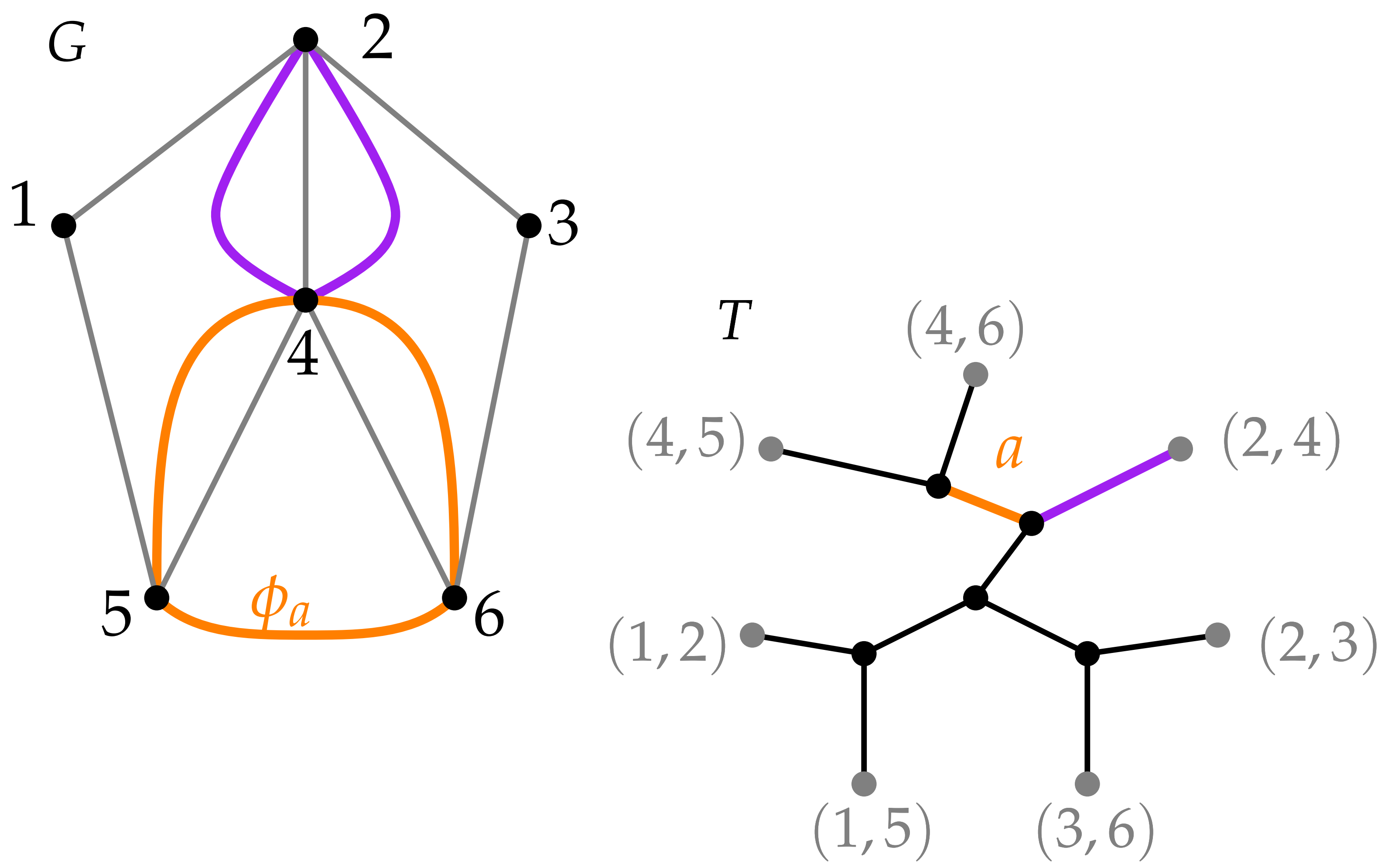

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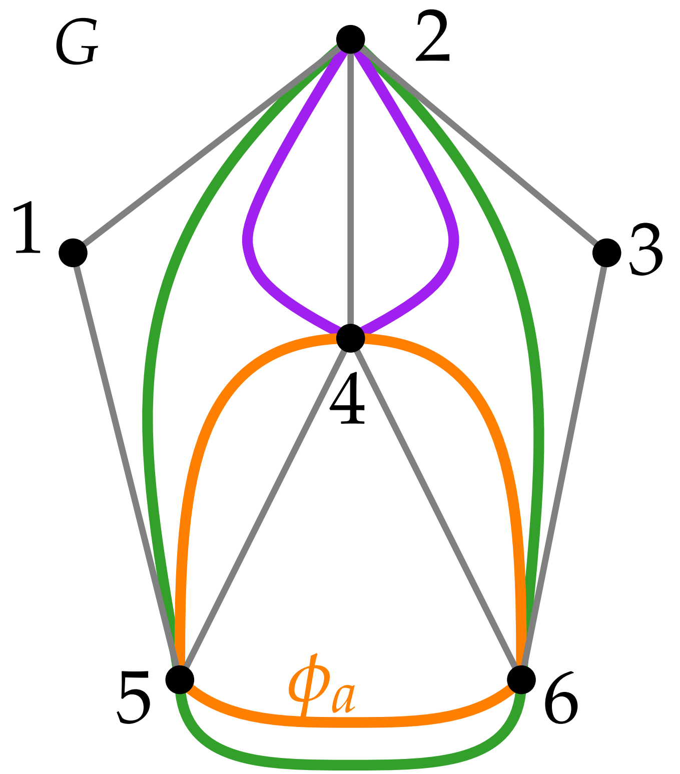

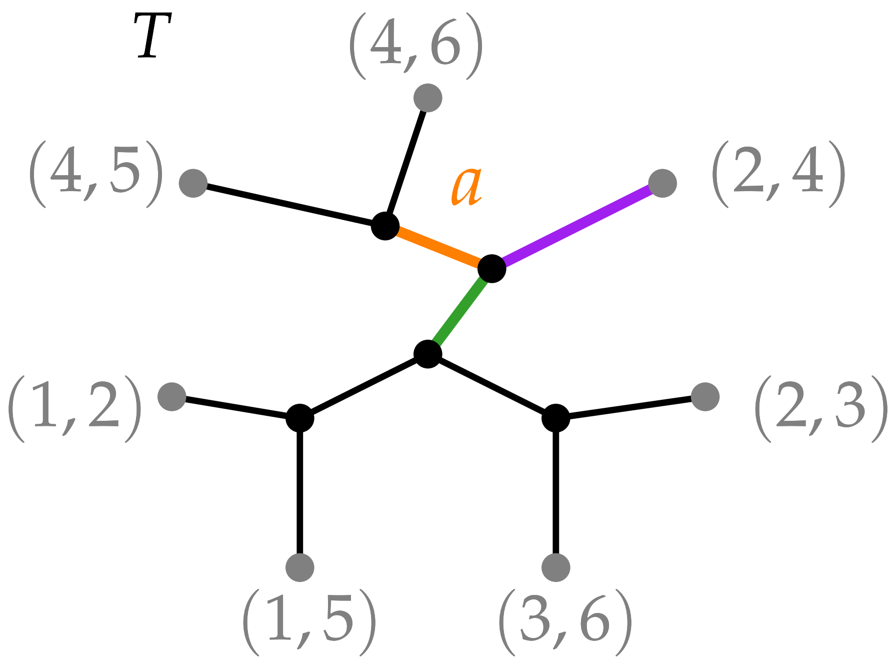

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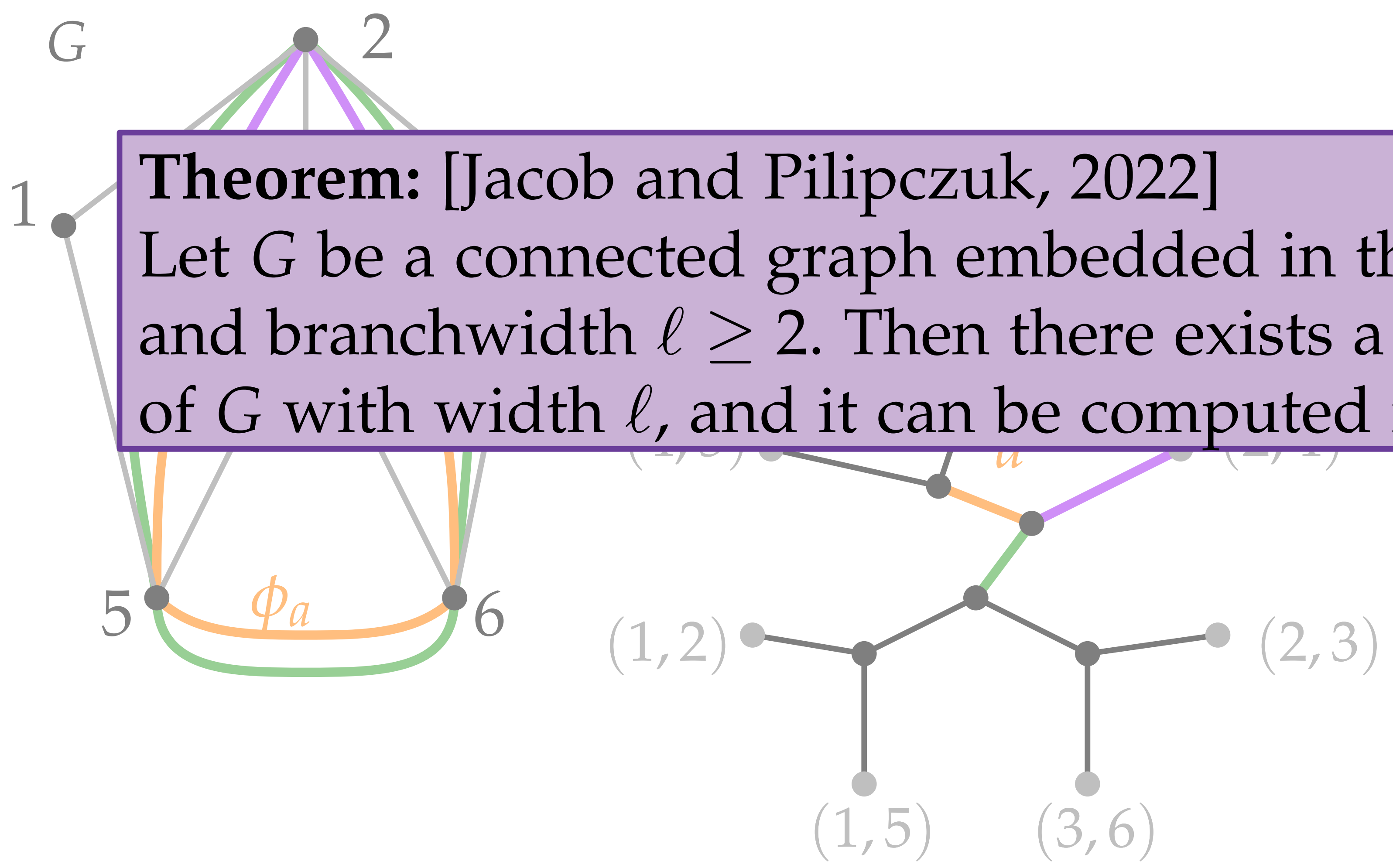

Theorem: [Jacob and Pilipczuk, 2022]
Let G be a connected graph embedded in the sphere with n vertices and branchwidth $\ell \geq 2$. Then there exists a sphere-cut decomposition of G with width ℓ, and it can be computed in $\mathcal{O}(n^3)$ time.

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Parametrization by Branchwidth

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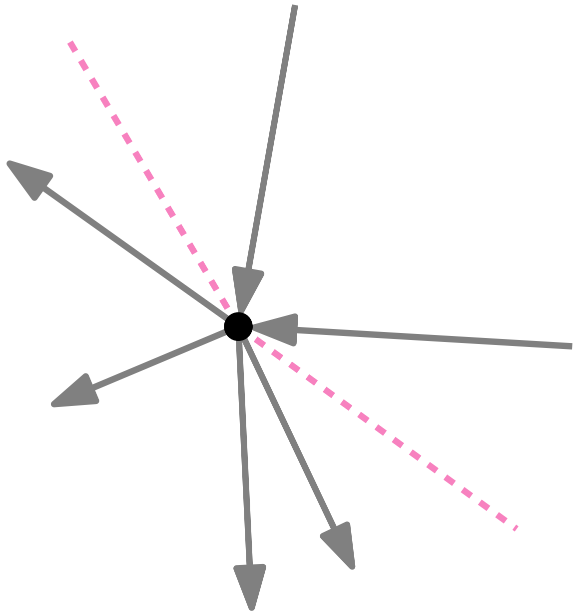

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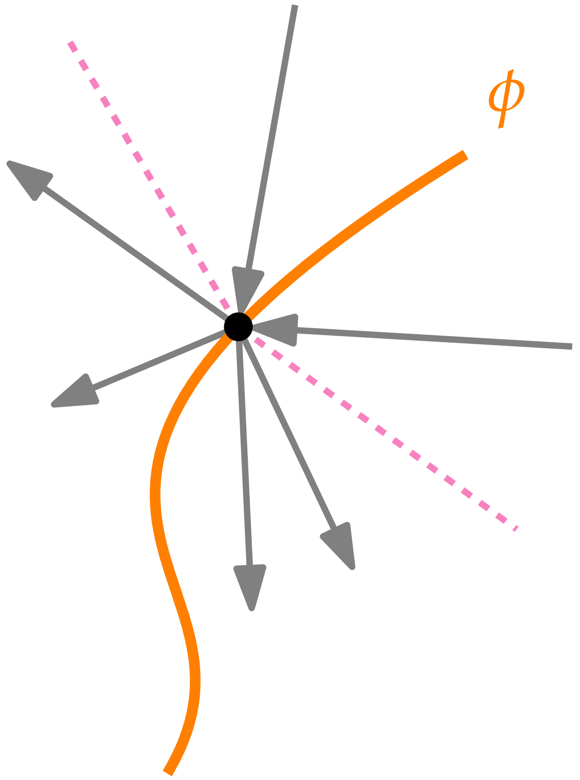

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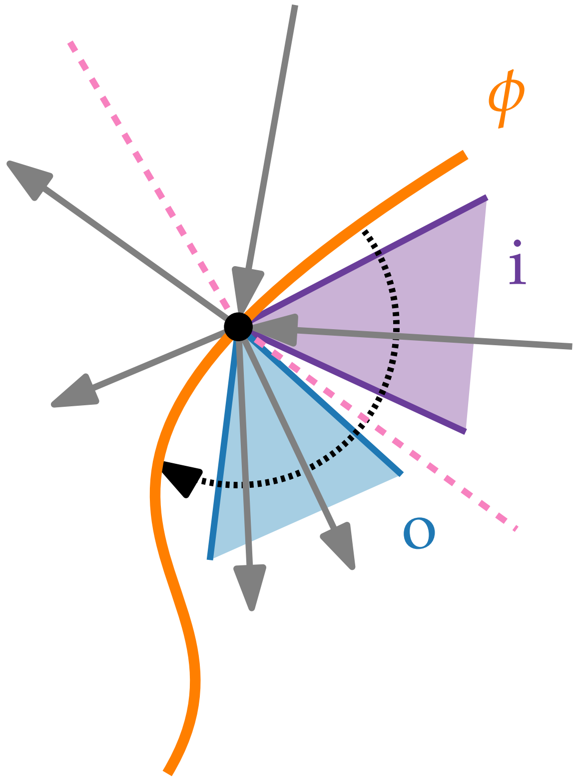

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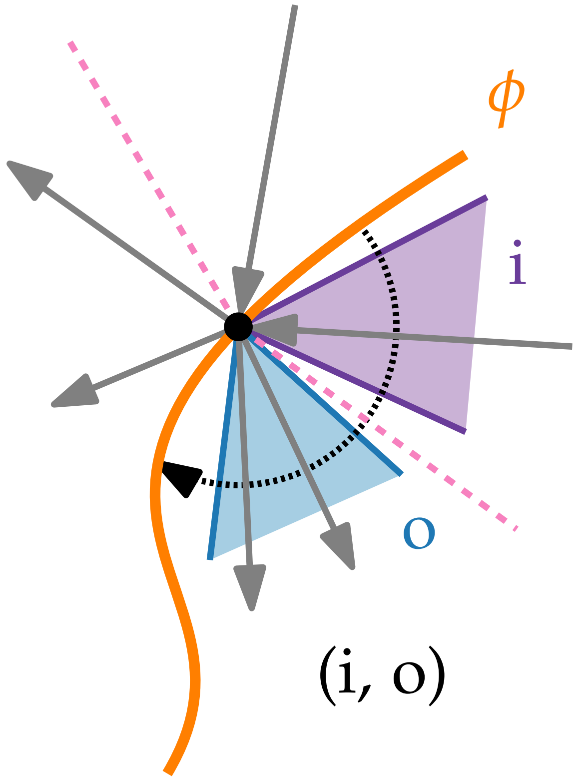

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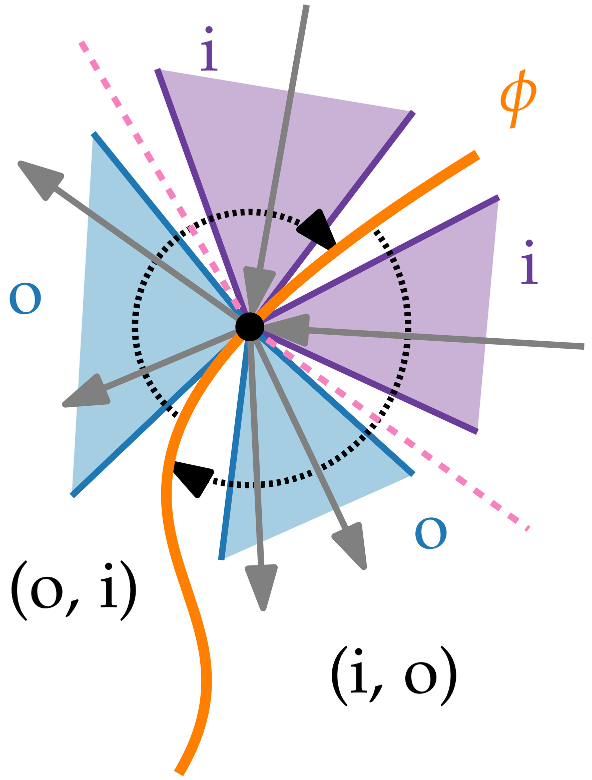

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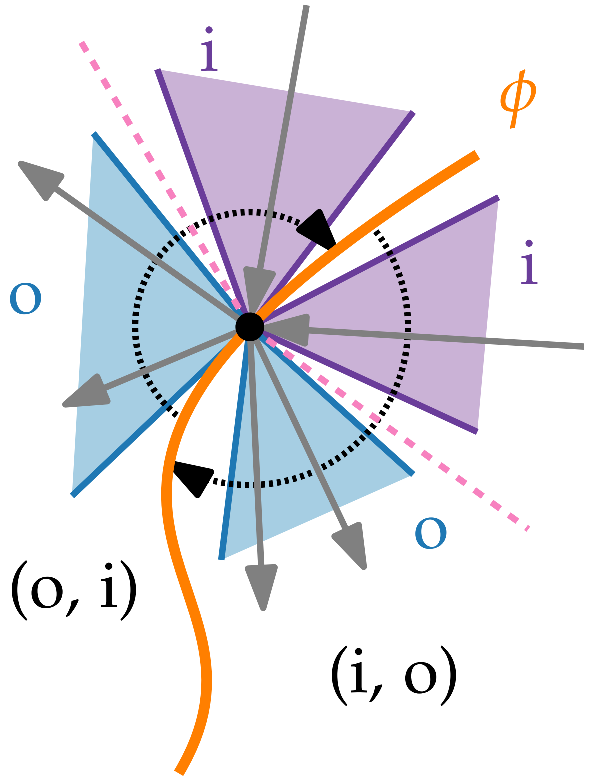

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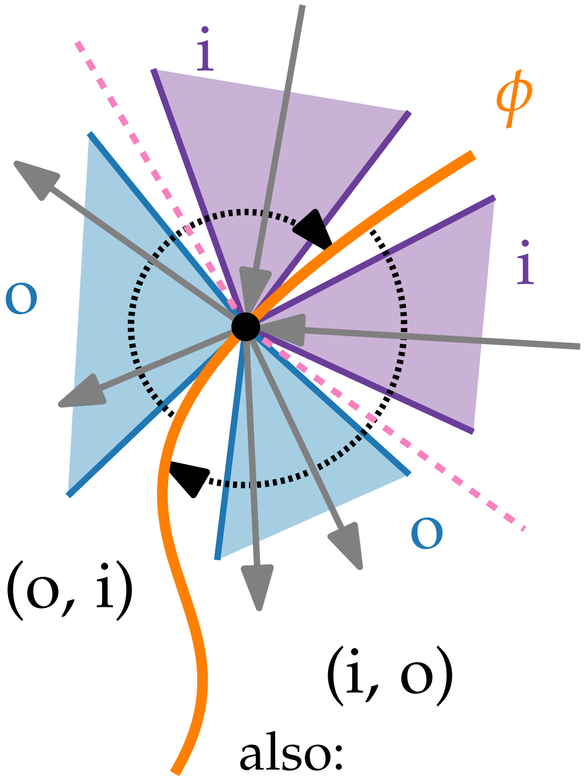

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- If v is bimodal, there are 6 possible configurations: (o), (i), (o, i), (i, o), (o, i, o), (i, o, i)
- Not unique:. E.g. (o) implies (i, o), (o, i), …

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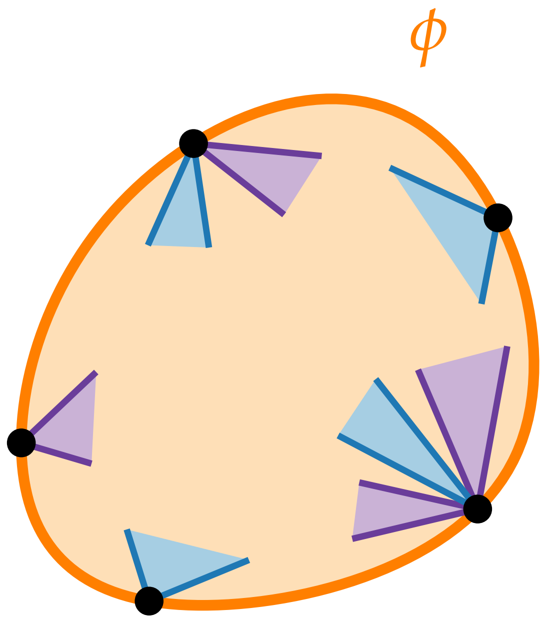

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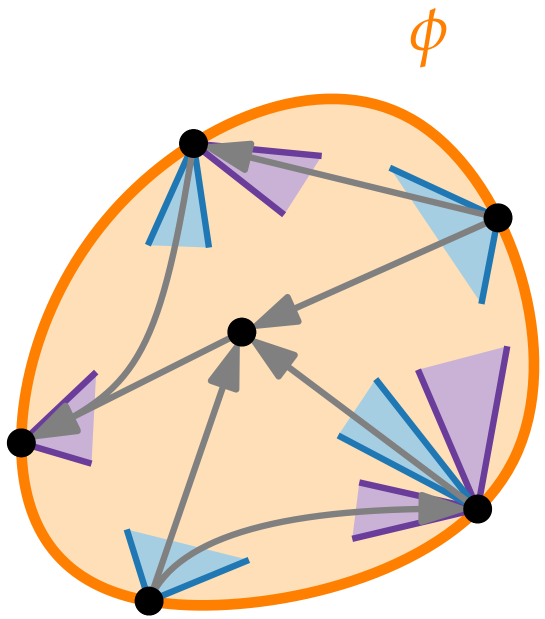

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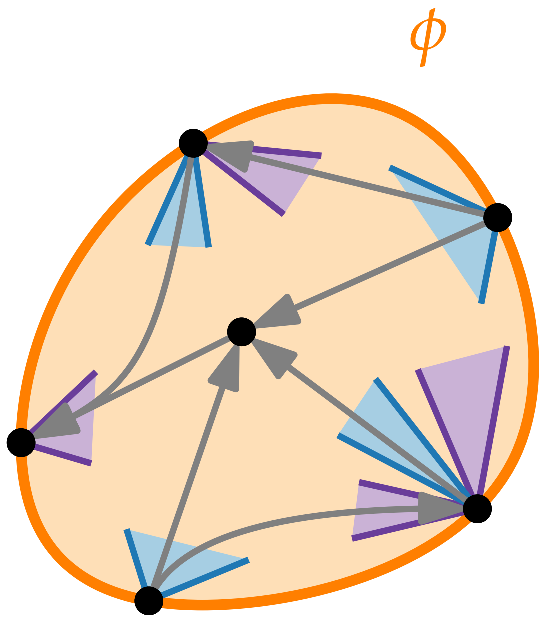

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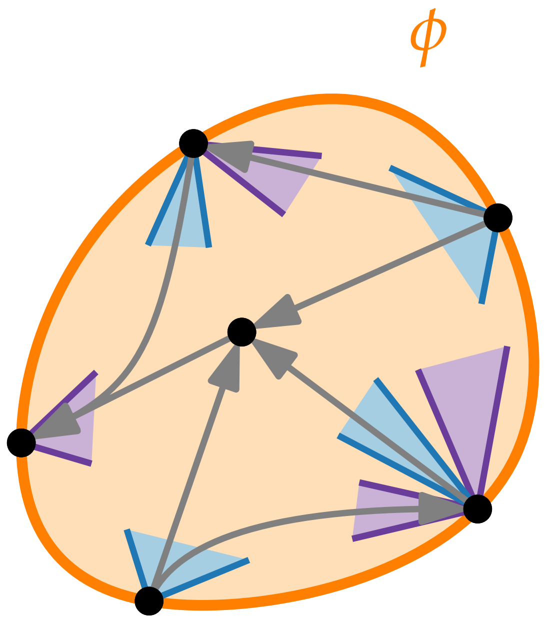

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  $\rightarrow$ There exist at most $6^{\mathrm{bw}(G)}$ configuration sets for ϕ.

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- Two configurations X, X' are compatible, if their concatenation – after deleting consecutive duplicates – is a substring of (o, i, o) or (i, o, i).

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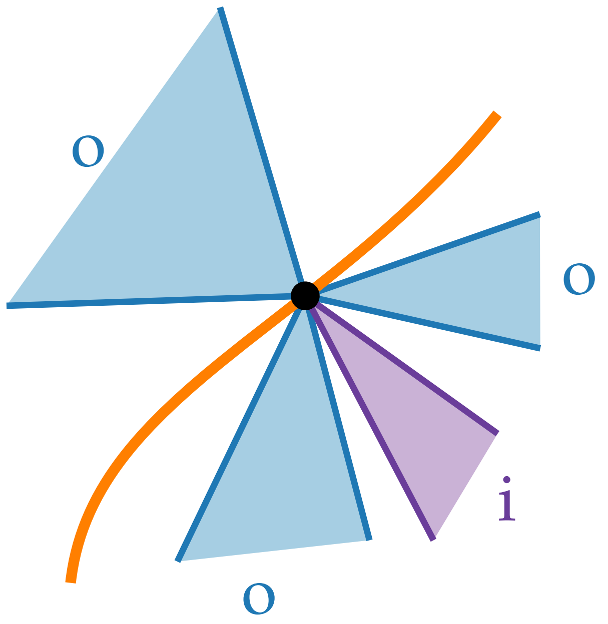

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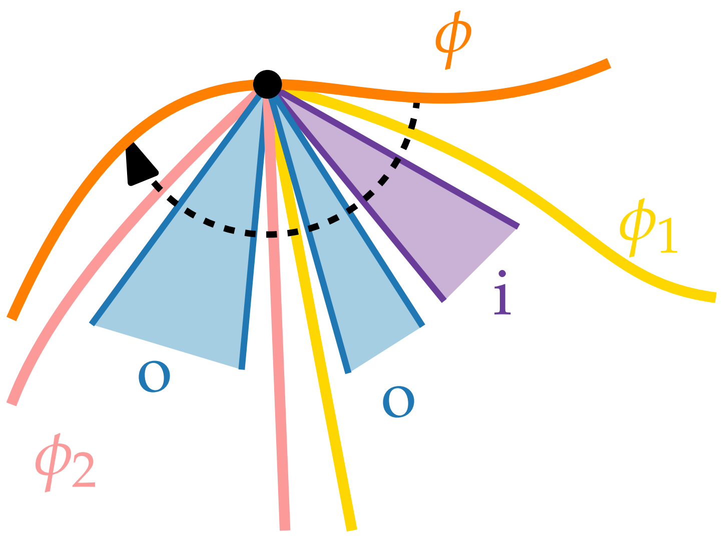

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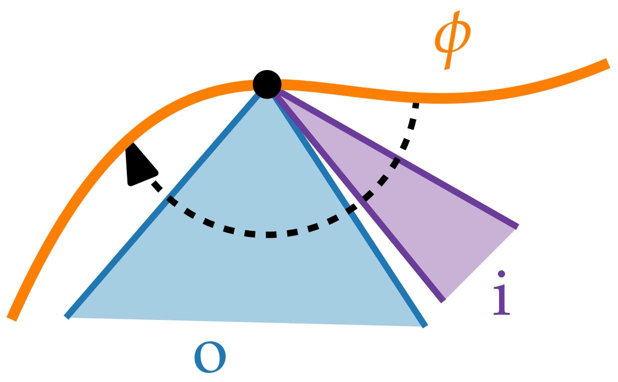

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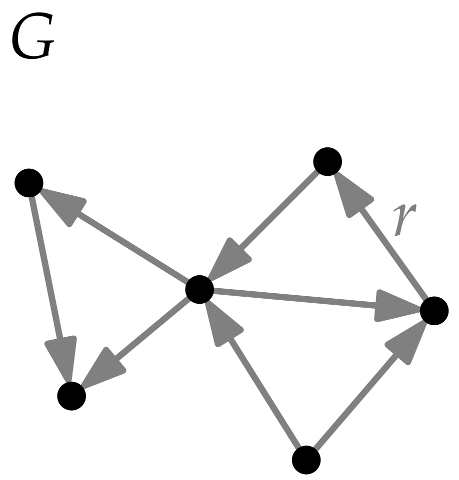

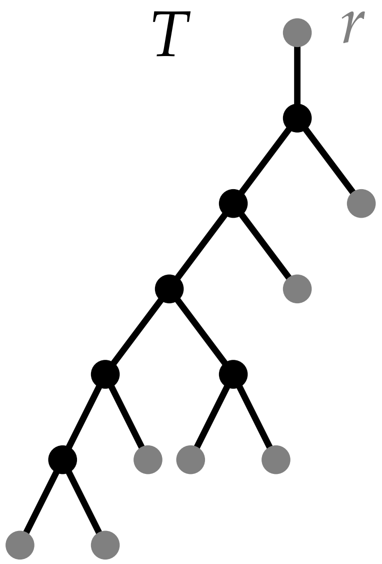

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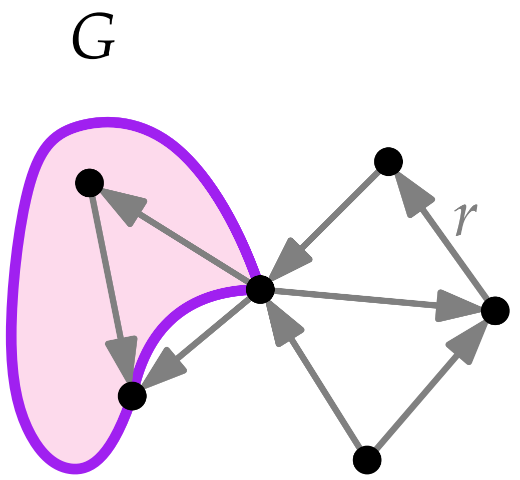

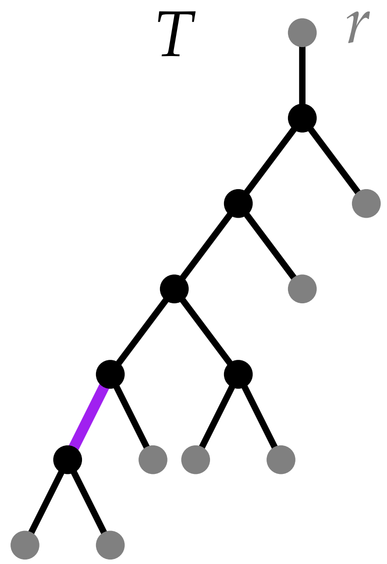

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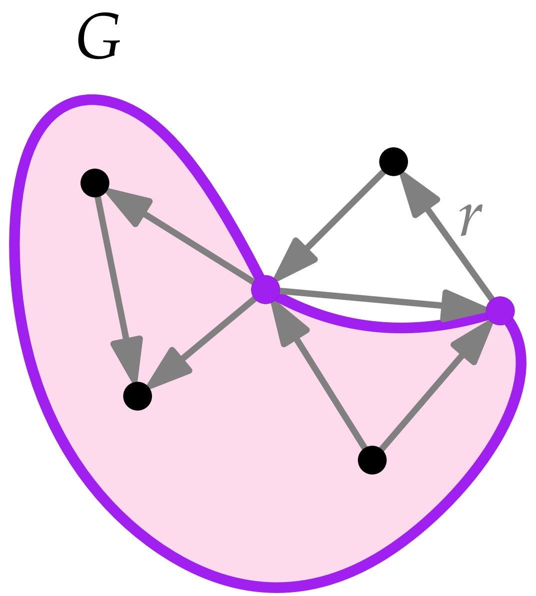

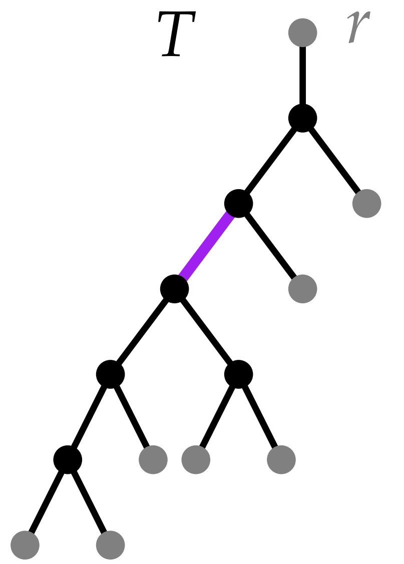

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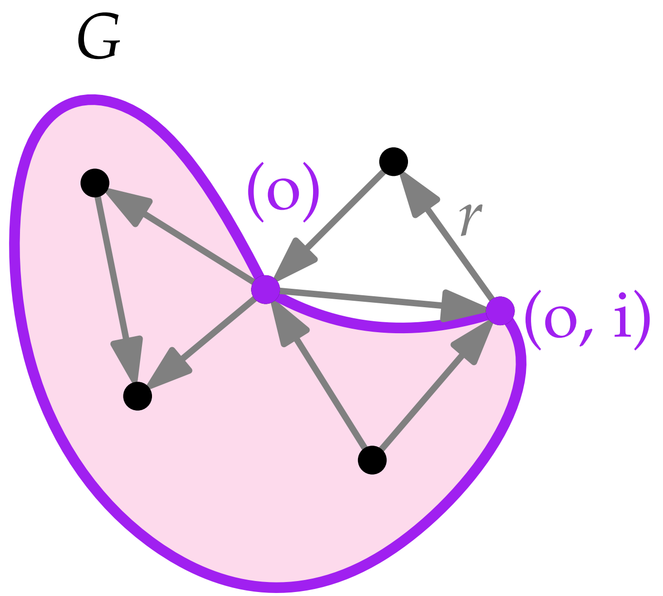

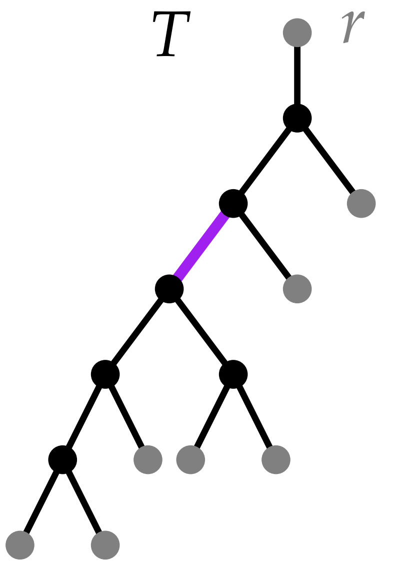

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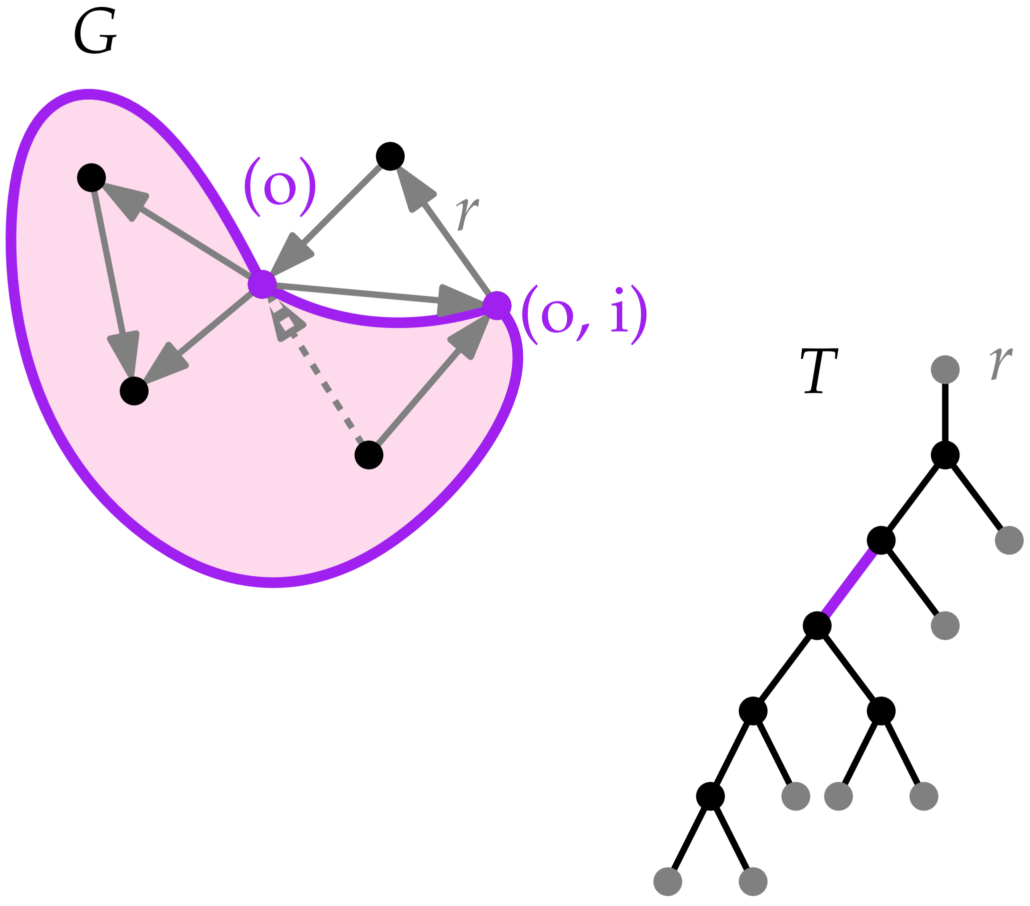

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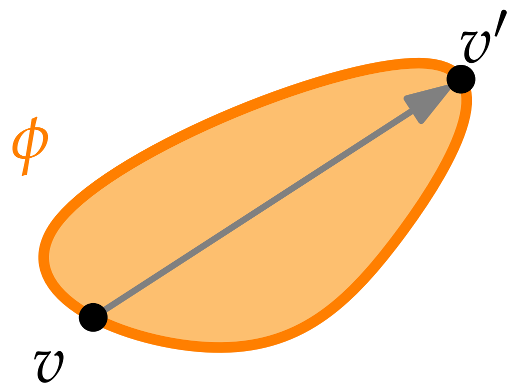
- Base Case: The curve ϕ contains a single edge $e = (v, v')$.



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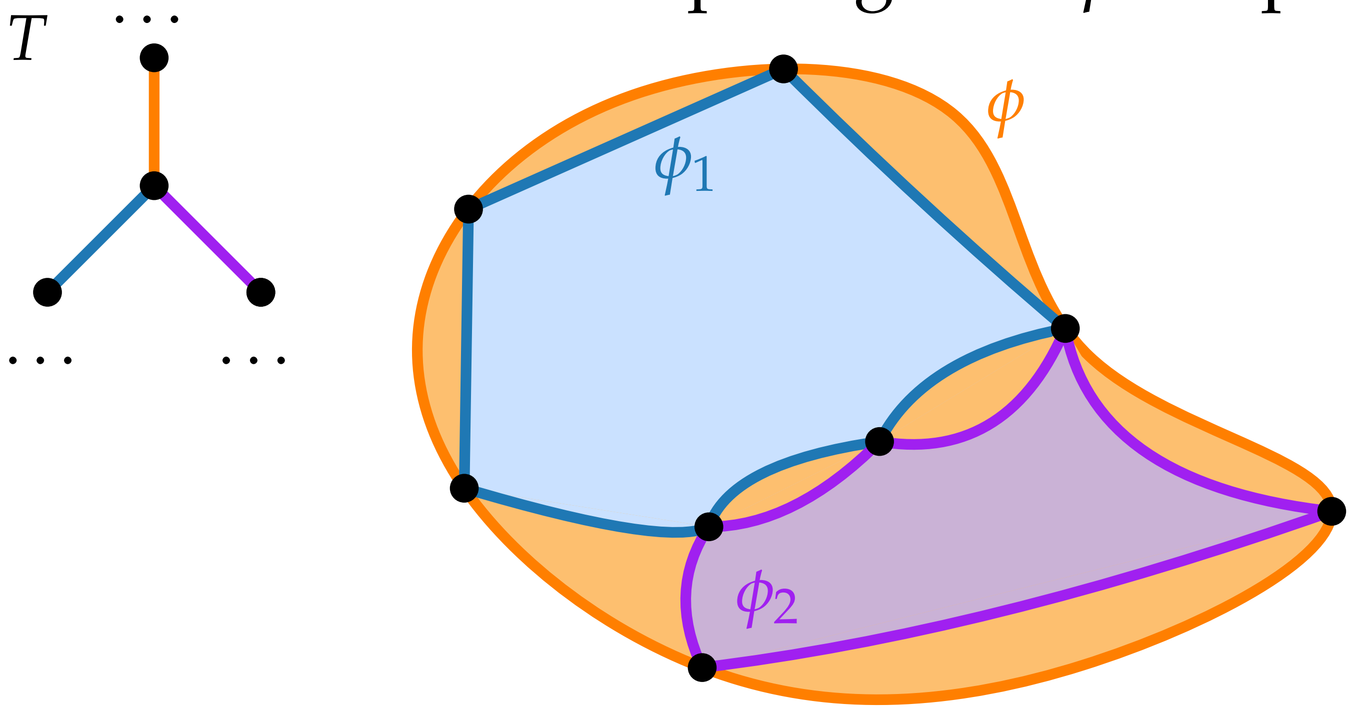
Proof sketch:

- Inductive Step: edges in ϕ are partitioned by ϕ_1, ϕ_2



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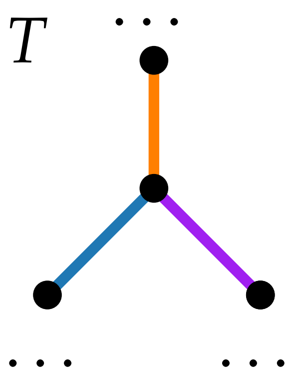

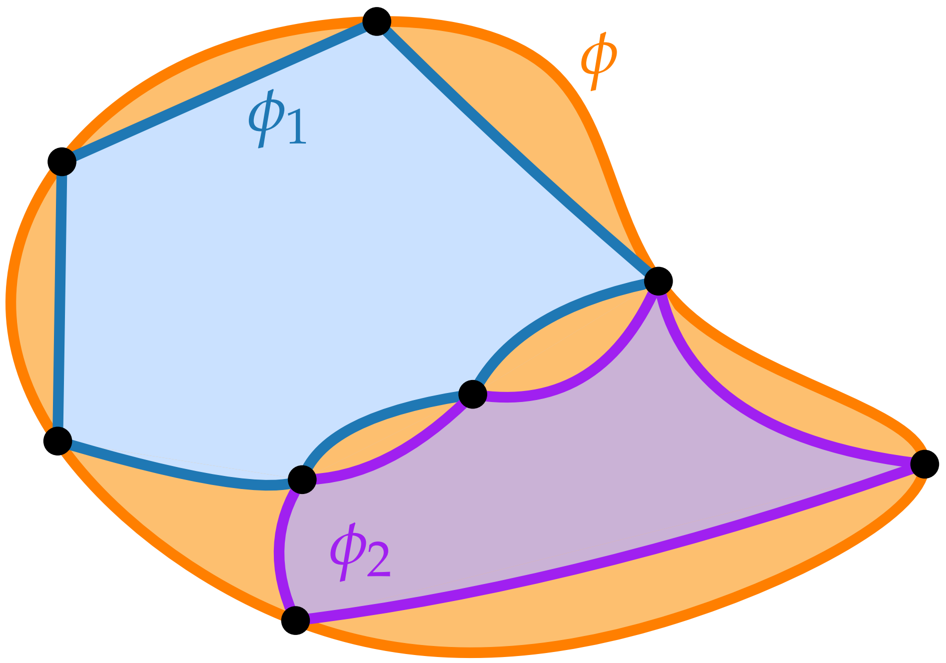

Iterate through every combination of configuration sets $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2$ for the curve ϕ, ϕ_1, ϕ_2.

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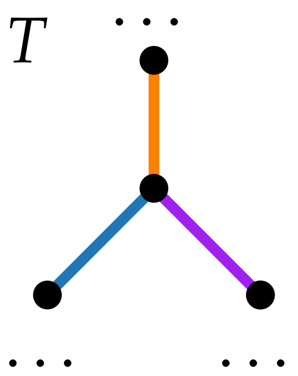
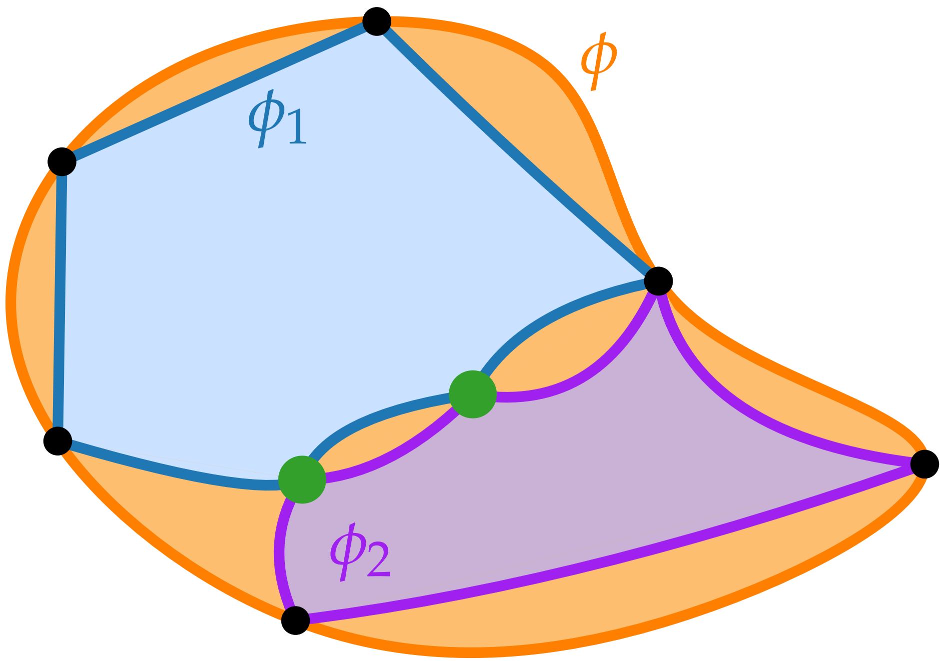
- Inductive Step: edges in ϕ are partitioned by ϕ_1, ϕ_2





Test for every vertex v that is cut by at least one of ϕ, ϕ_1, ϕ_2:

- If v is cut by ϕ_1 and ϕ_2, but not ϕ:

  Are $X_{v,1}, X_{v,2}$ compatible?

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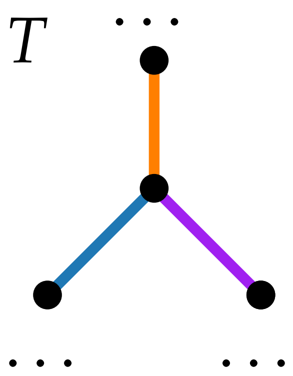
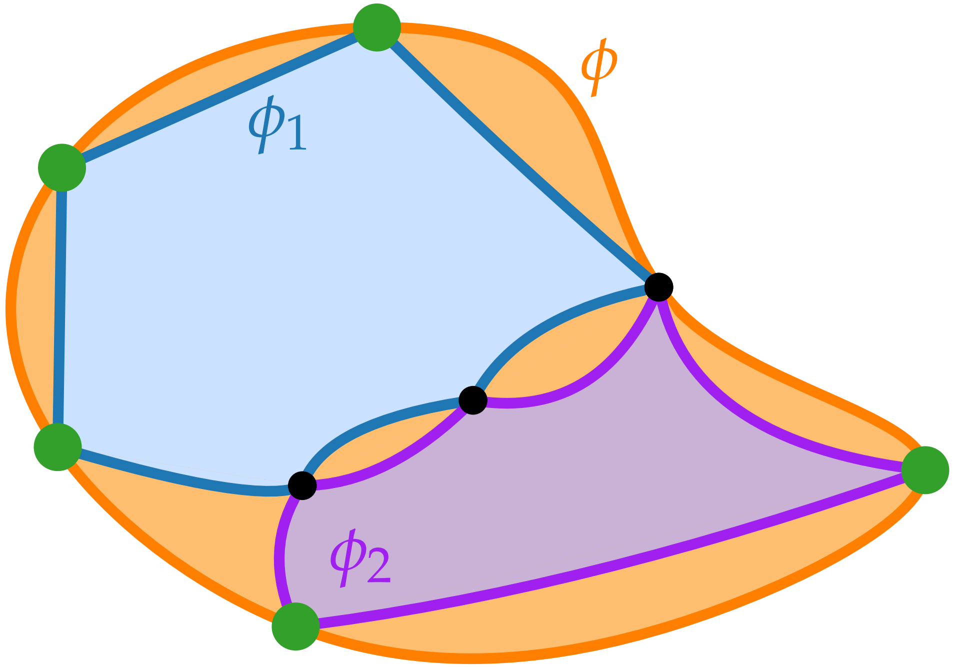




Test for every vertex v that is cut by at least one of ϕ, ϕ_1, ϕ_2:

- If v is cut by ϕ and only one of ϕ_1, ϕ_2:

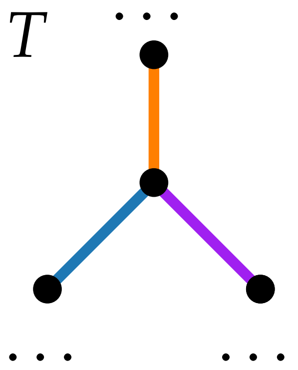
  Is $X_{v,1}$ (or $X_{v,2}$) a substring of X_v?

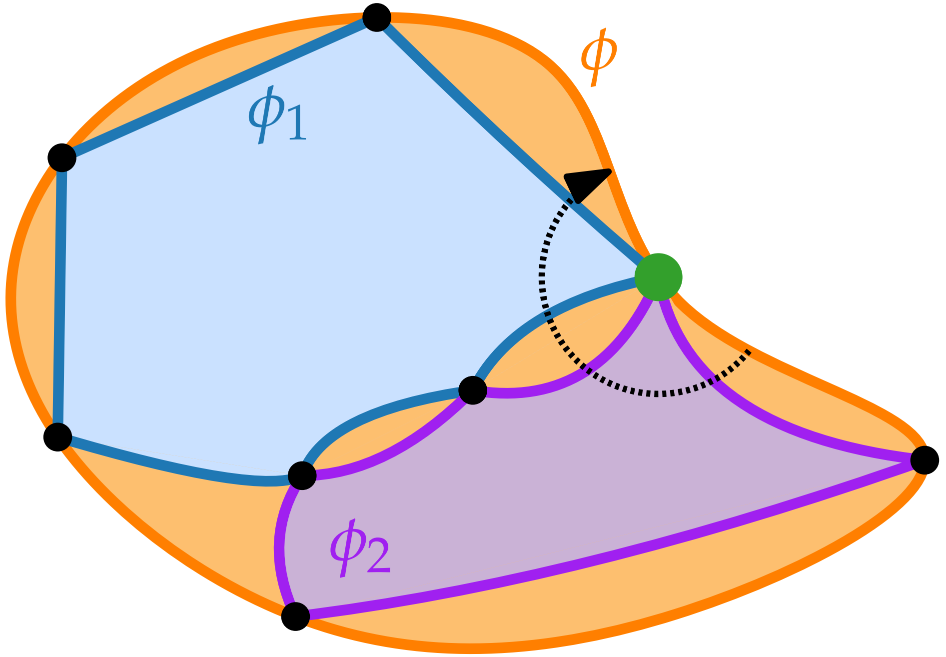
Parametrization by Branchwidth: Proof Sketch

Theorem 1:
There is an algorithm that solves MWBS in $2^{\mathcal{O}(\text{bw}(G))} \cdot n^{\mathcal{O}(1)}$ time. In particular, MWBS is FPT if parameterized by branchwidth.

Proof sketch:

- Inductive Step: edges in ϕ are partitioned by ϕ_1, ϕ_2





Test for every vertex v that is cut by at least one of ϕ, ϕ_1, ϕ_2:

- If v is cut by all three of ϕ, ϕ_1, ϕ_2:

  Are $\mathcal{X}_{v,2}$ and $\mathcal{X}_{v,1}$ compatible with respect to $\mathcal{X}_v$?

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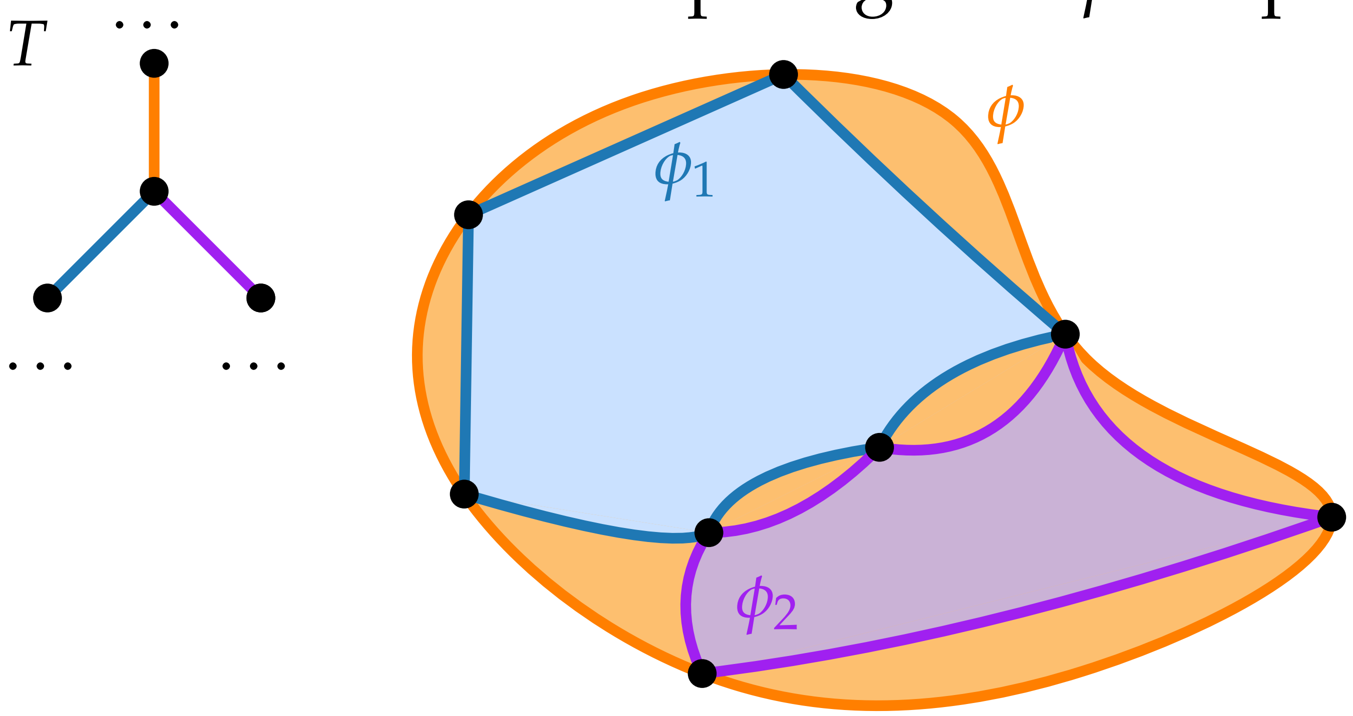

Runtime for one step:
$\mathcal{O}(6^{3\cdot \text{bw}(G)}) \cdot n^{\mathcal{O}(1)} = 2^{\mathcal{O}(\text{bw}(G))} \cdot n^{\mathcal{O}(1)}$.

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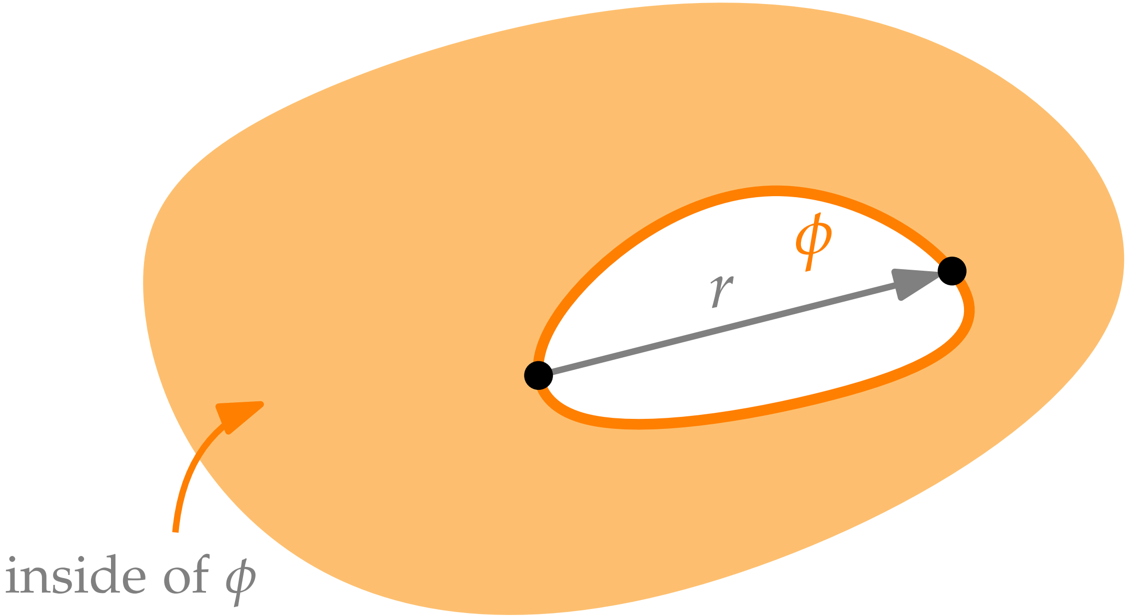
- Final Step: only root-edge left



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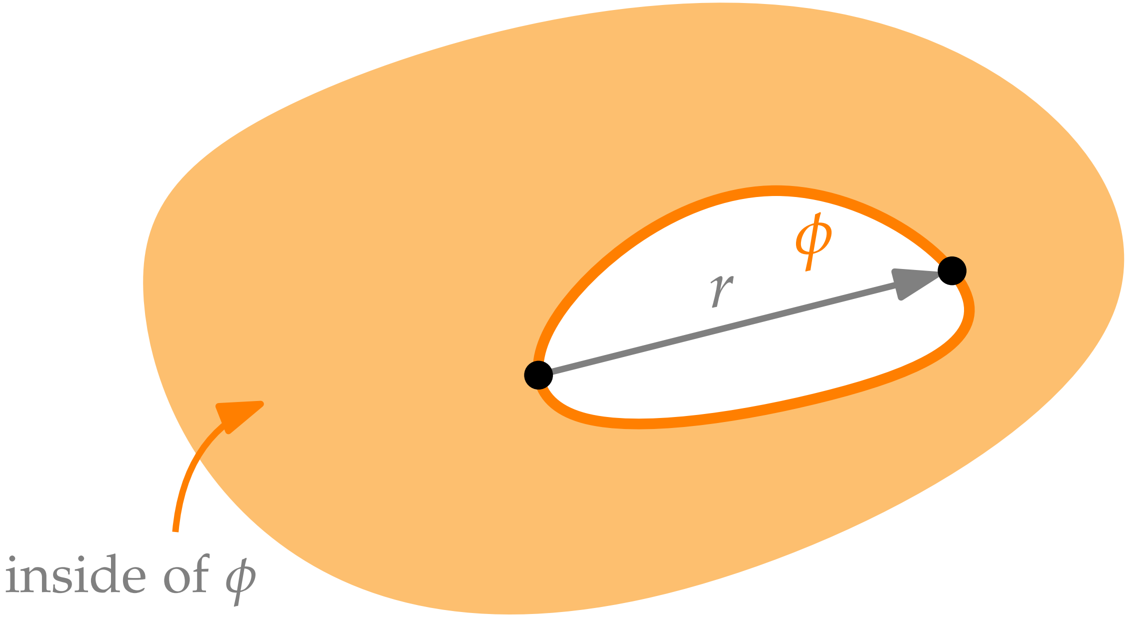

□

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Corollary 2:
There is an algorithm that solves MWBS in $2^{\mathcal{O}(\sqrt{n})}$ time.

Parametrization by the Number of Non-Bimodal Vertices

Theorem 2:
There exists an algorithm that solves MWBS with b non-bimodal vertices in $2^{\mathcal{O}(\sqrt{b})} \cdot n^{\mathcal{O}(1)}$ time. In particular, MWBS is FPT if parameterized by b.

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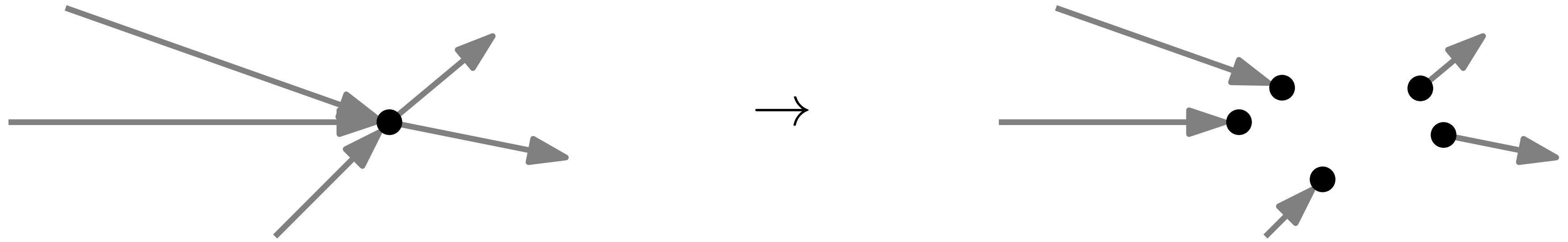

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$\rightarrow$ At most b vertices with degree ≥ 2.

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Running time: $2^{\mathcal{O}(\mathrm{bw(G')})} \cdot n^{\mathcal{O}(1)}$

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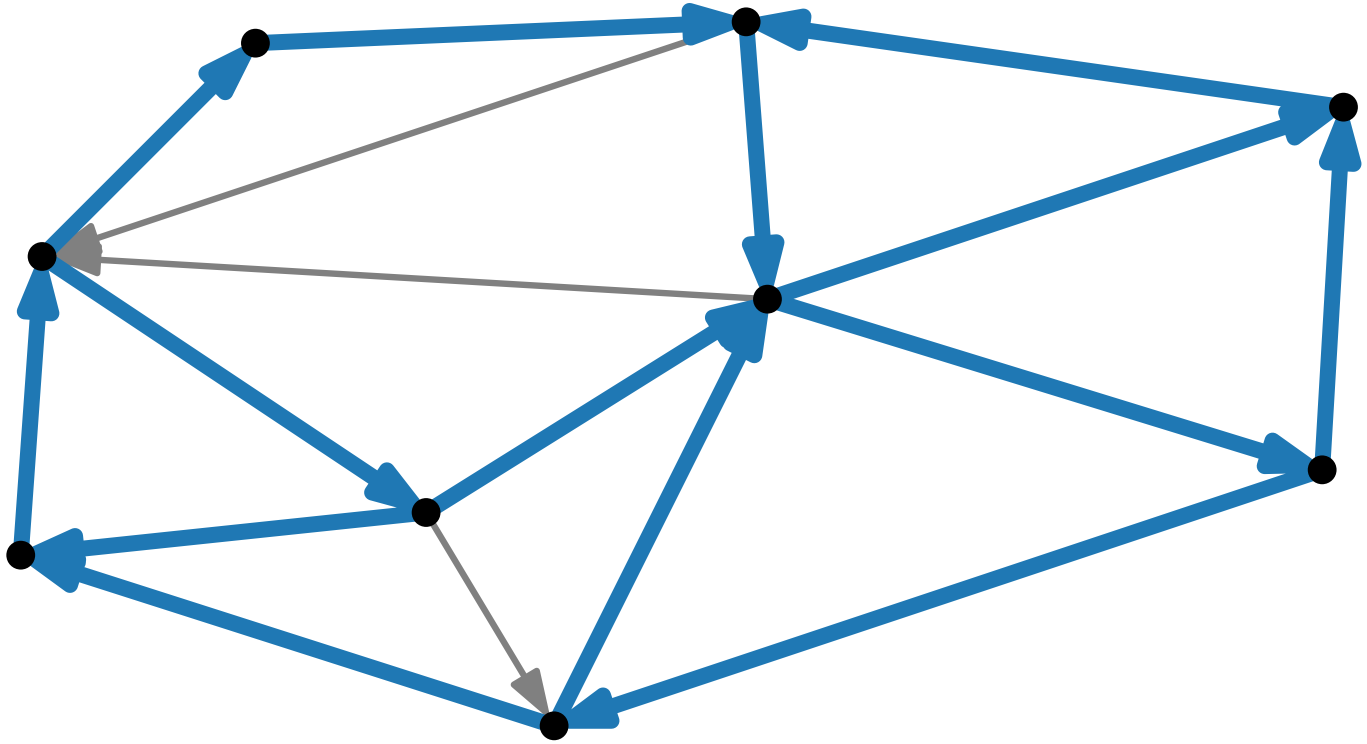
□

Open Problems

- Extend to the maximum k-modal subgraph problem for any given even integer $k \geq 2$.



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- Limit the number of edges that can be deleted by an integer h.

  Possible parameters:
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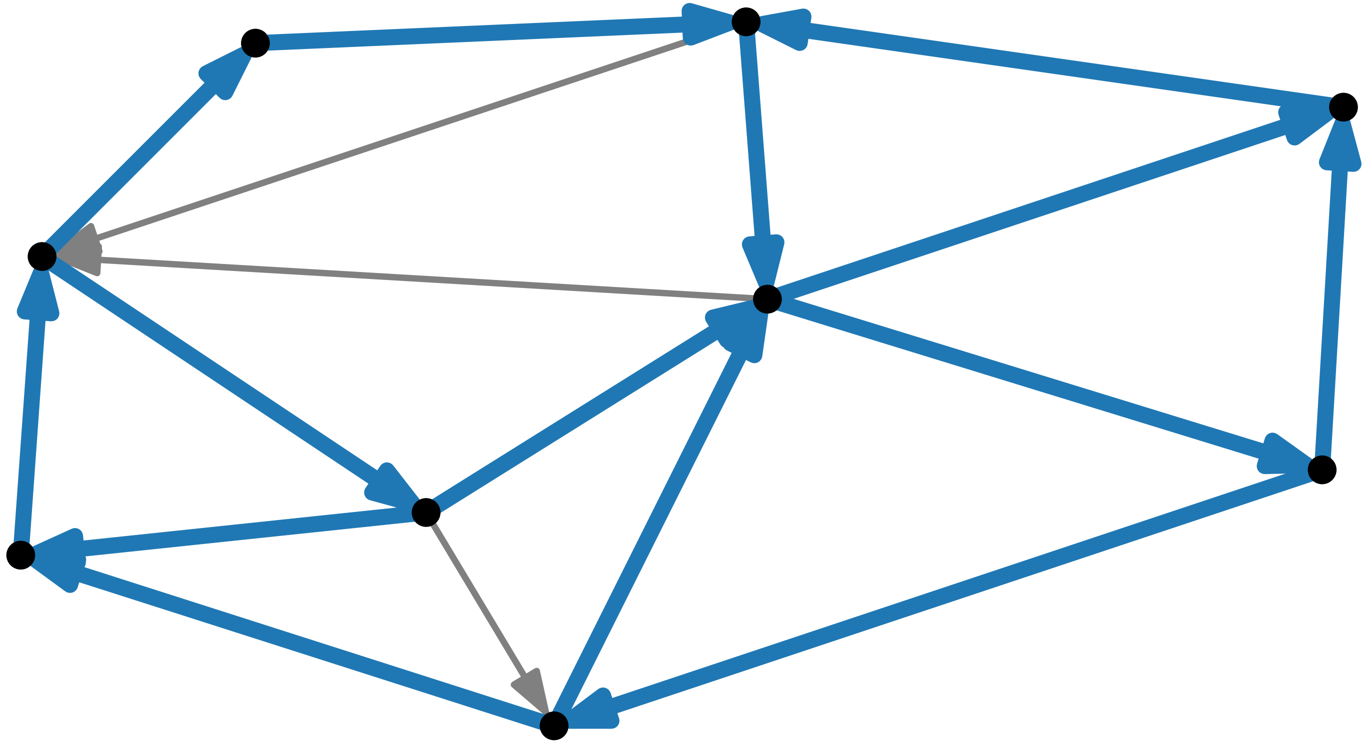

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- Study MBS in the variable embedding setting.