

# Upward and Orthogonal Planarity are $W[1]$ -hard by Treewidth

Bart M. P. Jansen, **Liana Khazaliya**, Philipp Kindermann,  
Giuseppe Liotta, Fabrizio Montecchiani, Kirill Simonov

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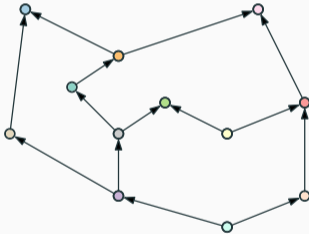
September 22, 2023

# What we are interested in

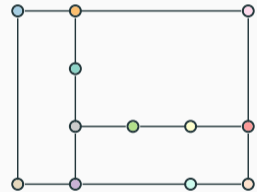
Directed graph  $\vec{G}$



Upward planar drawing



Orthogonal drawing



# Upward/Orthogonal Planarity Testing

With fixed embedding:      poly-time solvable

With variable embedding:    NP-complete

For the variable embedding:  $n^{O(tw)}$ -algorithms

Orthogonal: [GD 2019, E. Di Giacomo, G. Liotta, F. Montecchiani]

Upward: [SoCG 2022, S. Chaplick et al.]

Question: [SoCG 2022, S. Chaplick et al.]

Is Upward Planarity  $W[1]$ -hard of FPT when parameterized by  $tw$ ?

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Our Main Result:

Both Upward and Orthogonal Planarity testing are  $W[1]$ -hard.

# Upward/Orthogonal Planarity Testing

With fixed embedding:           poly-time solvable

With variable embedding:       NP-complete

For the variable embedding:  $n^{O(tw)}$ -algorithms

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Our Main Result:

Known  $n^{O(tw)}$ -algorithms cannot be improved to  $n^{o(tw)}$  under ETH.

# Overview [Key steps]

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# Outline

- Multicolored Clique

bounded pw



- All-or-Nothing Flow

- All-or-Nothing Flow on Planar graphs



$pw^\theta$  is  $\mathcal{O}(pw)$

- Circulating Orientation on Planar graphs

pw of the triangulation



- Orthogonal/Upward Planarity Testing

+ Concluding Remarks



# Multicolored Clique to All-or-Nothing Flow

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# Multicolored Clique (MClique)

Multicolored Clique

Input: An undirected simple graph  $G$  and a partition of its vertex set into  $k$  sets  $V_1, \dots, V_k$ , each consisting of  $N$  vertices.

Parameter:  $k$ .

Question: Does  $G$  contain a clique  $C \subseteq V(G)$  such that  $|C \cap V_i| = 1$  for each  $i \in [k]$ ?

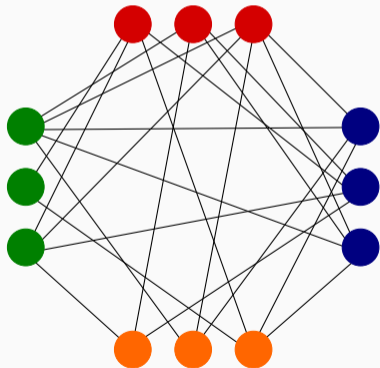
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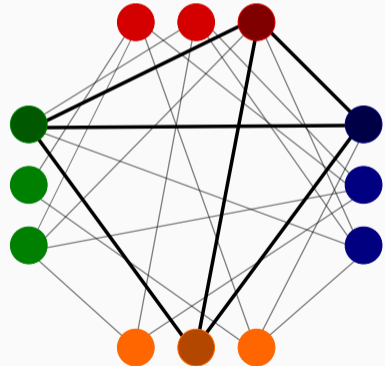
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# All-or-Nothing Flow<sup>1</sup> (AoNF)

All or Nothing Flow

Input: A flow network  $(G, c, s, t)$  and a positive integer  $\mathcal{F}$ .

Question: Does there exist an  $st$ -flow of value exactly  $\mathcal{F}$ , such that the flow through any arc  $uv \in E(G)$  is either 0 or equal to  $c(uv)$ ?

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<sup>1</sup>XNLP (at least  $W[1]$ -hard) when parameterized by  $\text{tw}$ : H. L. Bodlaender et al.  
Problems Hard for Treewidth but Easy for Stable Gonality, WG'22

# All-or-Nothing Flow (AoNF)

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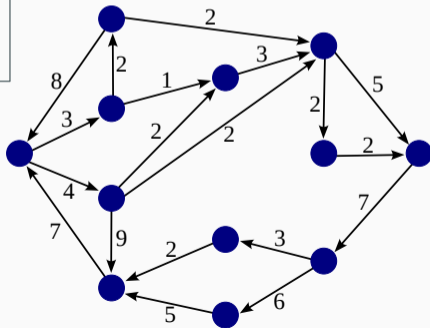
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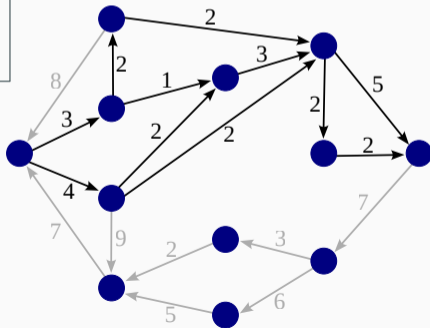
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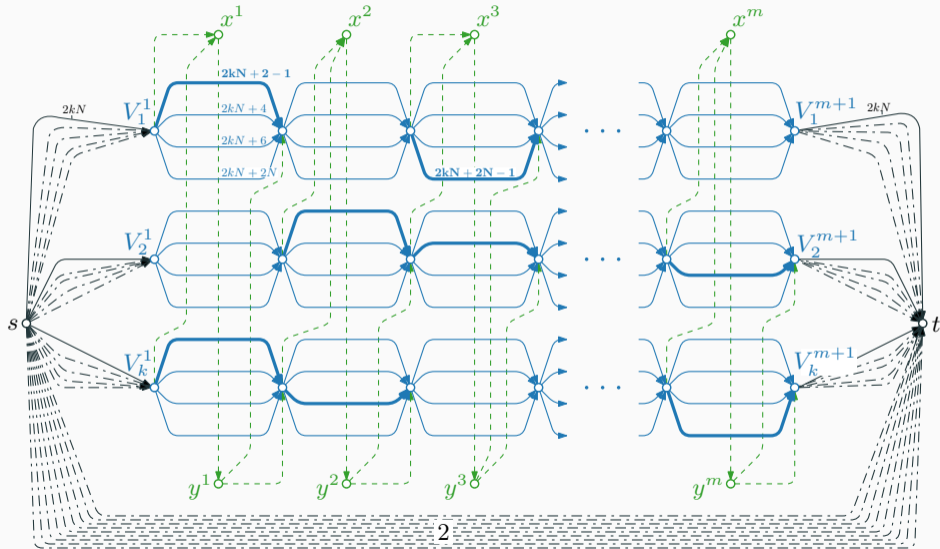
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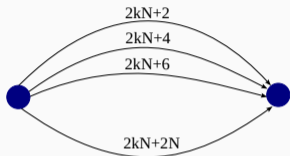
# AoNF: $(G', c, s, t)$ and $\mathcal{F} = k(2kN + 2N)$



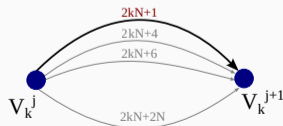
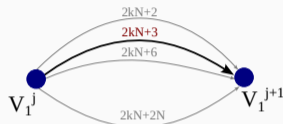


# MClique: $(G, (V_1, V_2, \dots, V_k)), |V_i| = N$

$$V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,N}\}.$$



Non-edge  $v_{1,2}v_{k,1}$  of  $G$ .

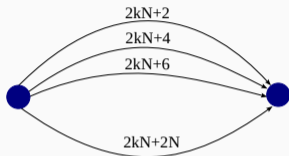


Inflow  $\in [2kN + 2, 2kN + 2N]$ ;

Inflow is even.

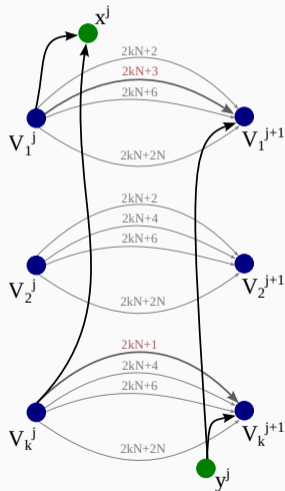
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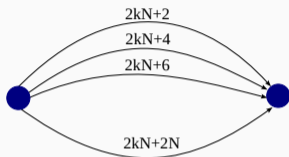
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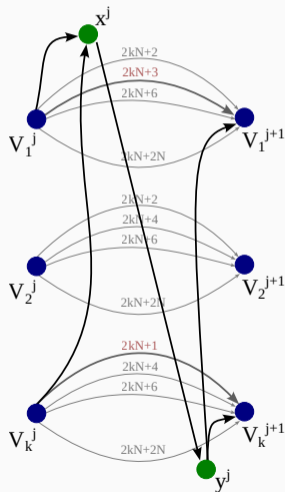
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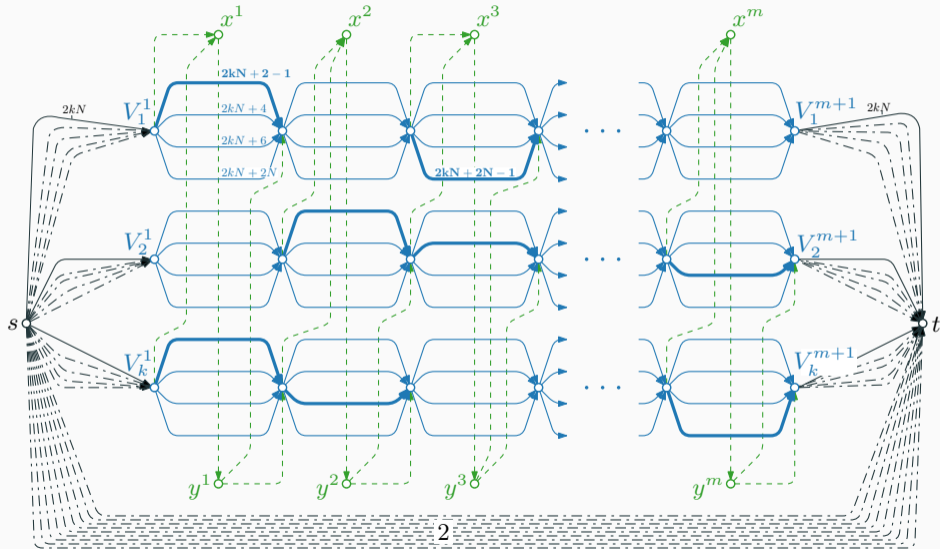


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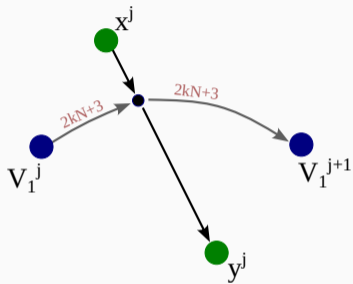
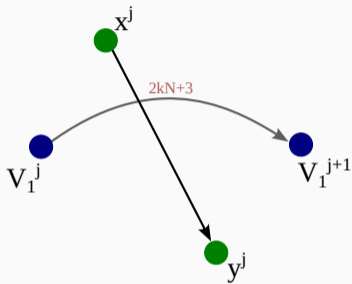
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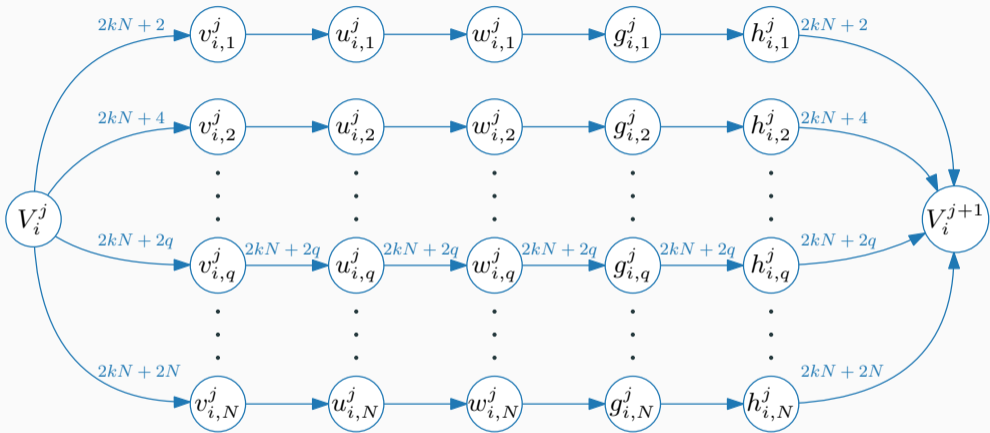
# Planarization of the AoNF

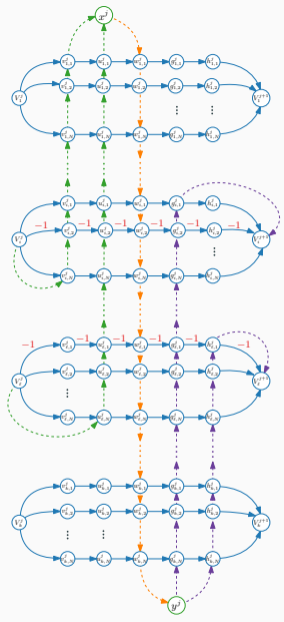
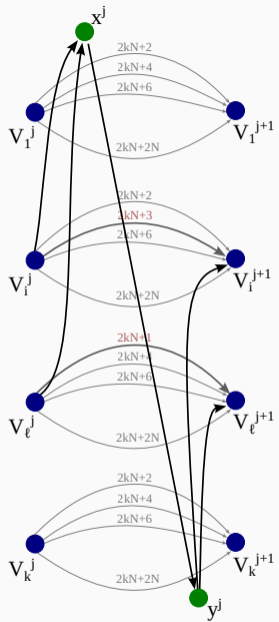
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# Observation



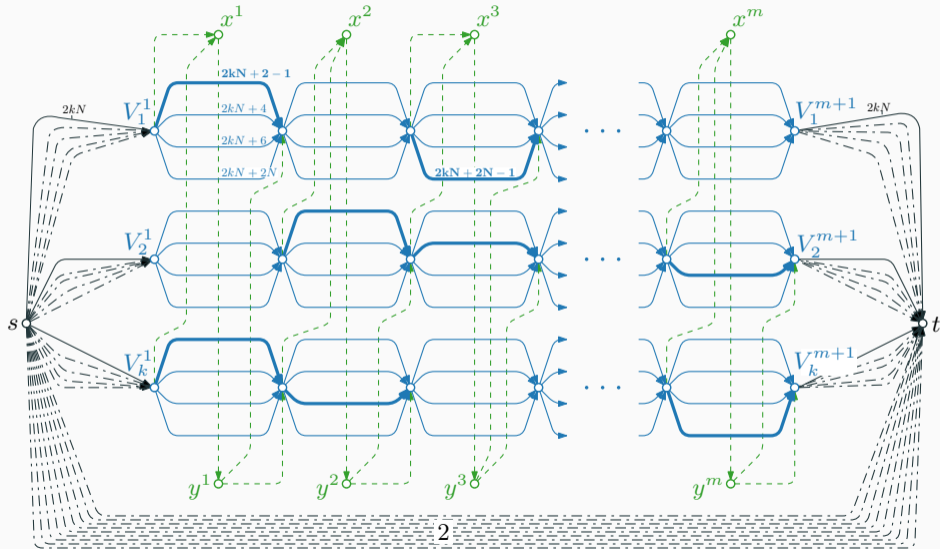
Planarizing a crossing of two edges via a degree-4 vertex does not change the answer, when the capacities of the edges differ.



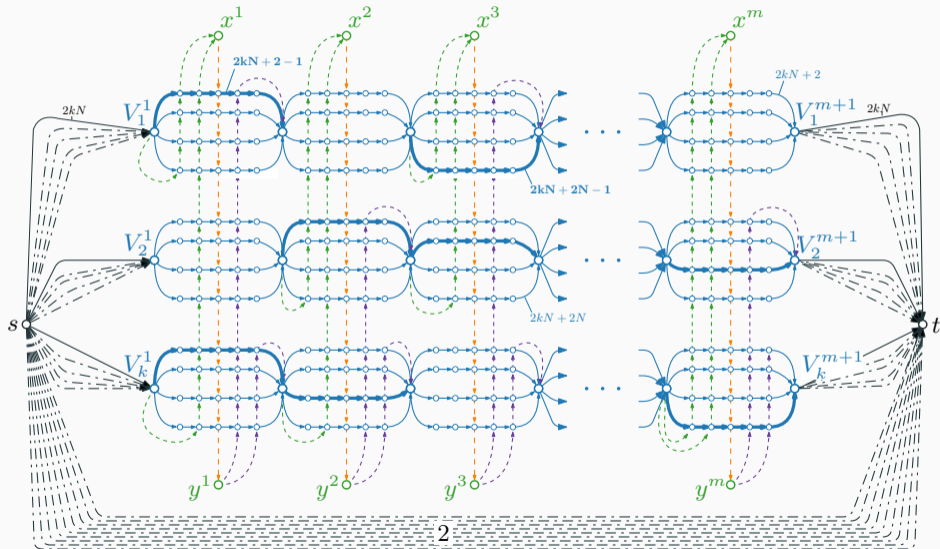




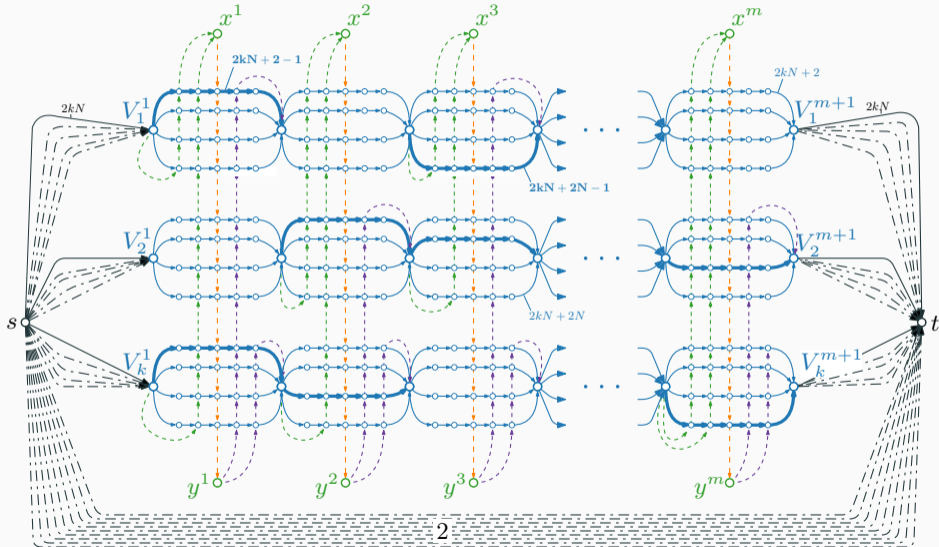
# AoNF: $(G', c, s, t)$ and $\mathcal{F} = k(2kN + 2N)$



# Planar AoNF: $(G'', c, s, t)$ and $\mathcal{F} = k(2kN + 2N)$



# First remark: bounded pathwidth



# All-or-Nothing Flow (planar) to Circulating Orientation

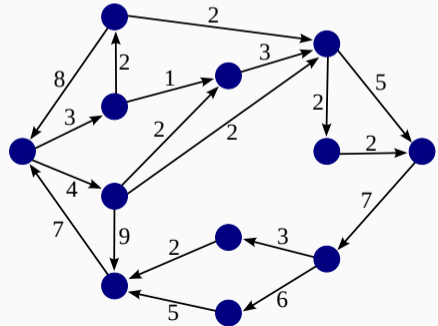
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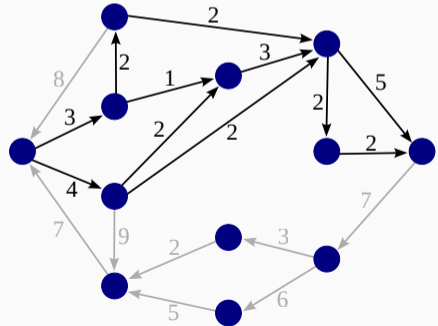


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# Circulating Orientation (CO)

## Circulating Orientation

Input: An undirected graph  $G$  with an edge-capacity function  $c: E(G) \rightarrow \mathbb{Z}_{\geq 0}$ .

Question: Is it possible to orient the edges of  $G$ , such that for each vertex  $v \in V(G)$  the total capacity of edges oriented into  $v$  is equal to the total capacity of edges oriented out of  $v$ ? (Such an orientation is called a circulating orientation.)

# Circulating Orientation (CO)

## All or Nothing Flow

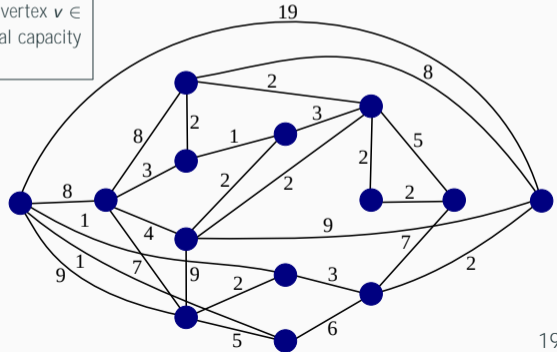
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## Circulating Orientation

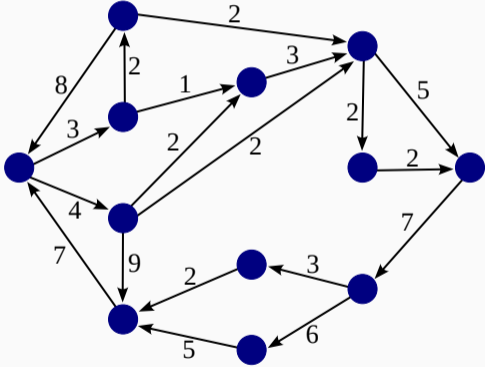
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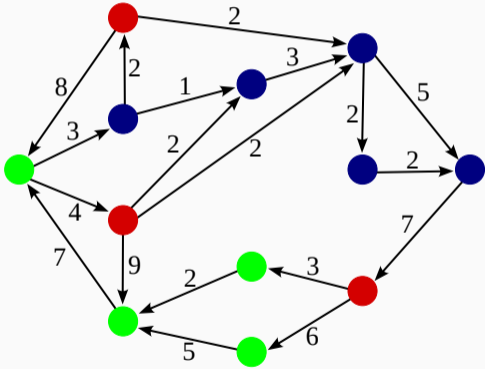




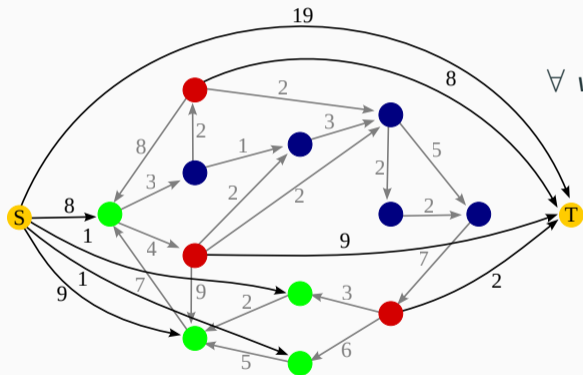
# AoNF to CO



# AoNF to CO

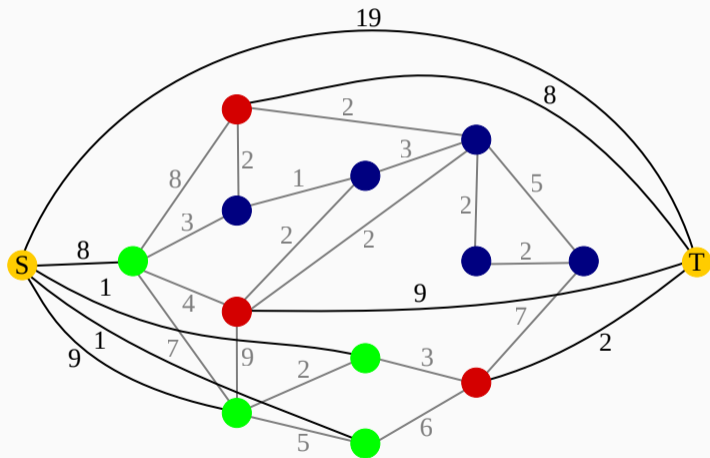


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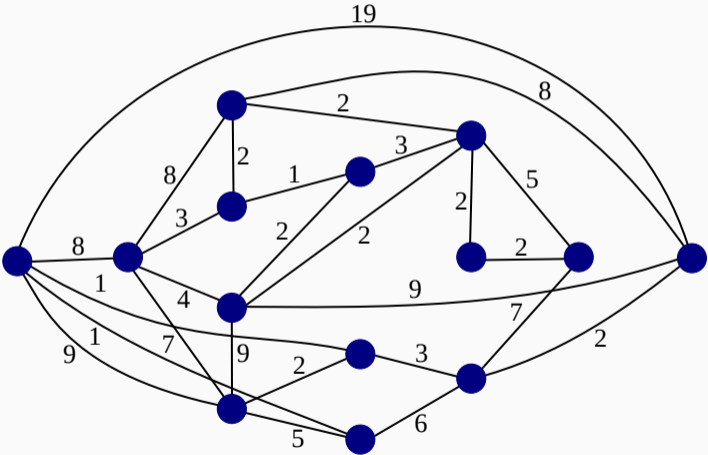


$\forall v \in V(G) \setminus \{s, t\}: d_G^+(v) = d_G(v)$ ,  
the source has no incoming arcs,  
the sink has no outgoing arcs,  
and  $\mathcal{F} = d_G^+(s)/2 = d_G(t)/2$ .

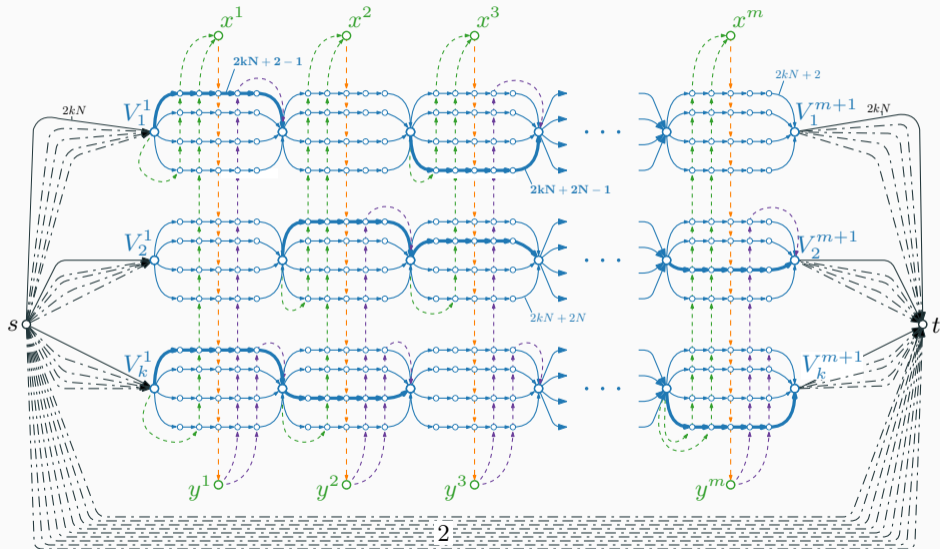
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# Second remark: a nice embedding



# Circulating Orientation to Upward Planarity Testing

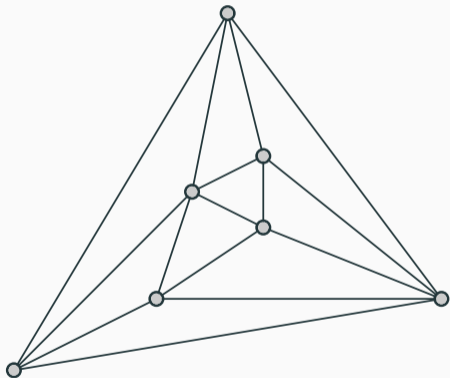
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Theorem (Biedl'16)

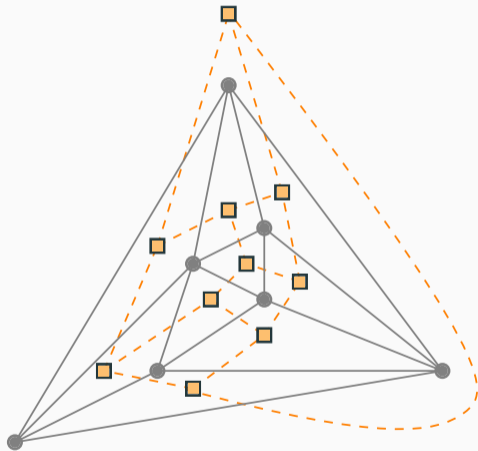
*There is a polynomial-time algorithm that, given a simple planar graph  $G$  of pathwidth  $k$  on at least three vertices, outputs a plane triangulation  $G^\theta$  of  $G$  such that  $\text{pw}(G^\theta) \in \mathcal{O}(k)$ .*



# Triangulated instance of CO



# Dual Graph

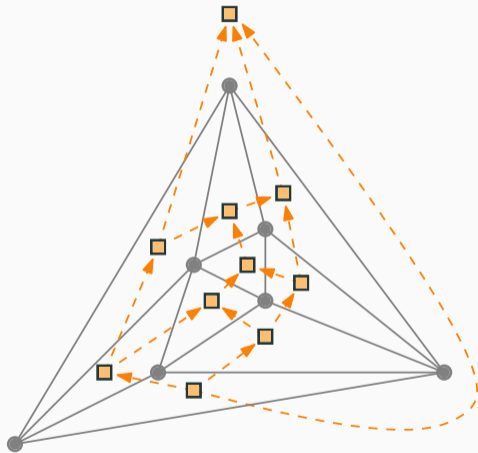


## Black Box #2

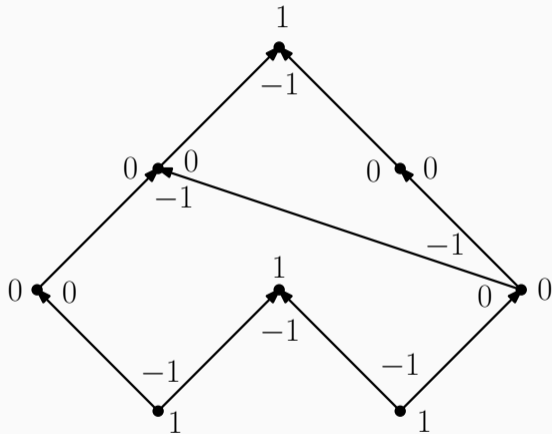
Theorem (Amini, Huc, and Pérennes'09)

*For a triconnected planar graph  $G$ ,  $\text{pw}(G^*) \leq 3 \text{pw}(G) + 2$ , where  $G^*$  is the dual graph of  $G$ .*

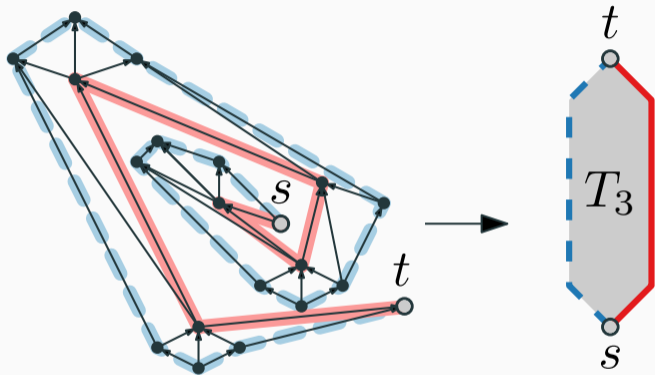
# Testing a Dual Graph for Upward Planarity



# Angle Assignment

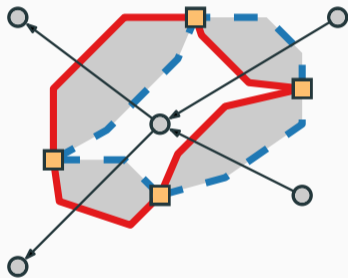
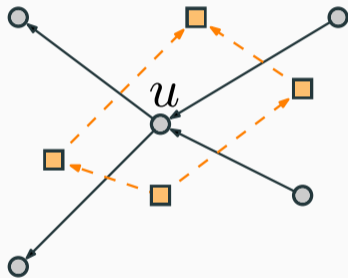


# Tendrils<sup>2</sup> Gadget



<sup>2</sup>A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

# Reduction Idea: Face Balancing



# ... and Orthogonal Planarity Testing

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# Differences

- Important that we start with a triangulated graph
- Subdivision of edges to allow an orthogonal embedding
- Orthogonal Tendril<sup>3</sup>

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<sup>3</sup>A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

## Concluding remarks

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We have proved that

Known  $n^{O(tw)}$ -algorithms cannot be improved to  $n^{o(tw)}$  under ETH.

What other points are also one might find interesting:

- Alternative<sup>4</sup> proof of NP-completeness
- Hardness extends for cutwidth of the primal

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<sup>4</sup>A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

# Further

- Membership in  $XNLP^5$  of both Upward and Orthogonal Planarity Testing: can be solved nondeterministically in time  $f(k)n^{O(1)}$  and space  $f(k)\log(n)$ ?
- FPT or  $W[1]$ -hard for taking as a parameter the cutwidth of the dual graph
- More restrictive parameterizations may yield FPT algorithms

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<sup>5</sup>H. L. Bodlaender et al. Parameterized Problems Complete for Nondeterministic FPT time and Logarithmic Space, FOCS'21

# Thanks for attention!

## Further directions

- Membership in XNLP
- Cutwidth of the dual graph
- Other parameterizations

## Contents

Overview [Key steps]  
MClique to AoNF  
Planar AoNF  
AoNF-pl to CO  
CO to UpPlanarity  
CO to OrtPlanarity  
Remarks