Minimizing the energy of spherical graph representations

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Graph Drawing 2023

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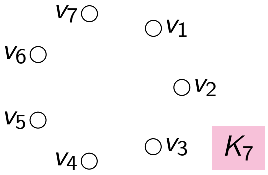
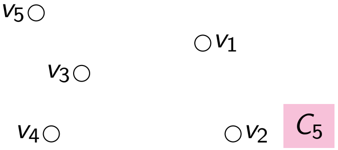
Graph Representations

Definition

A *representation of a graph G in $\mathbb{R}^d$* is a function $\mathbf{r} : V(G) \to \mathbb{R}^d$. The point $\mathbf{r}(v)$ is the *representation of $v \in V(G)$*.





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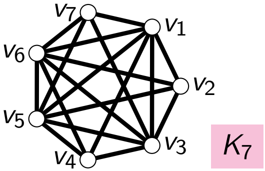
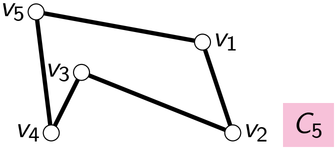
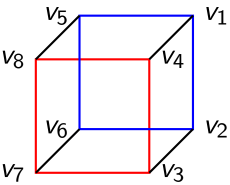
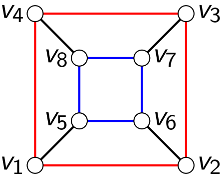


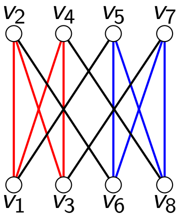


A representation induces a straight line/geometric drawing.

Graph Representations: Symmetry, Planarity, Partition







Graph Representations in $\mathbb{R}^d$

The *energy* of a representation $\mathbf{r} : V(G) \to \mathbb{R}^d$ of a graph G is defined as

$$\text{energy}(G, \mathbf{r}) = \sum_{uv \in E(G)} ||\mathbf{r}(u) - \mathbf{r}(v)||^2.$$

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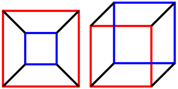
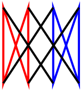
- energy minimization was used for plane graph drawings (Tutte) and for symmetrical representations (spectral method)

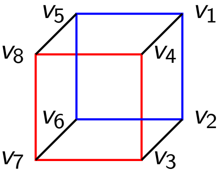


- energy maximization was used for graph partitions (Goemans-Williamson)



Definition

A *representation matrix* of a representation $\mathbf{r}$ of G in $\mathbb{R}^d$ is a $d \times v(G)$ matrix where column $v \in V(G)$ is $\mathbf{r}(v)$.



$$V = \begin{bmatrix} \mathbf{r}(v_1) & \mathbf{r}(v_2) & \mathbf{r}(v_3) & \mathbf{r}(v_4) & \mathbf{r}(v_5) & \mathbf{r}(v_6) & \mathbf{r}(v_7) & \mathbf{r}(v_8) \end{bmatrix}$$

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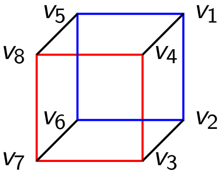
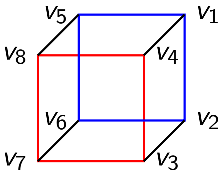


$$V = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Minimize the energy w.r.t

the rows of the representation matrix are orthogonal, unit vectors



$$V = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Algorithm 1: Spectral Graph Drawing Algorithm.

Input: Adjacency matrix A

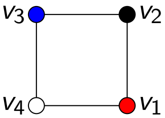
1. Compute an orthogonal, unit eigenvector basis for the $d + 1$ highest eigenvalues of A.
2. Let R be the representation matrix where the rows are the basis vectors.
3. $v \in V(G)$ is represented by column v of R.

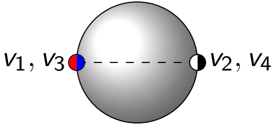
Drawings of platonic solids use λ_2-eigenspace.

Goemans, Williamson, 1995

1. Maximize the energy w.r.t

the vertices are on the unit d-sphere for some $d \geq 1$.





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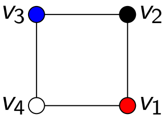

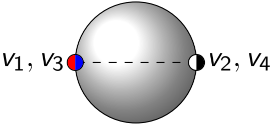

2. Partition vertices by taking a random hyperplane of the sphere.

Theorem (Goemans, Williamson, 1995)

The algorithm (1.+2.) produces an edge-cut with expected size at least 0.87856 times the size of the maximum edge cut of the graph.

Minimizing on the sphere

DeVos, Rogers, W., 2023

Minimize the energy w.r.t

$$\text{the vertices are on a unit d-sphere for some } d \geq 1 \qquad (1)$$

$$\text{the origin is the barycenter of the vertex representations} \qquad (2)$$

This minimization can be expressed via a semidefinite program (and hence approximated efficiently).

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A *unit barycentre* $\mathbf{0}$ representation is a representation which satisfies (1)+(2).

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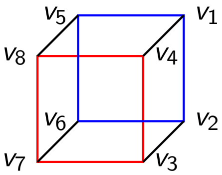

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Random Regular Graphs

Theorem (DeVos, Rogers, W., 2023)

For every random d-regular graph G there exists a minimum energy, unit barycentre $\mathbf{0}$ *representation* $\mathbf{r}$ *such that the inequalities*

$$(d-2\sqrt{d-1}-\epsilon)v(G) \leq \text{energy}(G,\mathbf{r}) \leq (d-2\sqrt{d-1}+\epsilon)v(G).$$

hold asymptotically almost surely for every $\epsilon > 0$.

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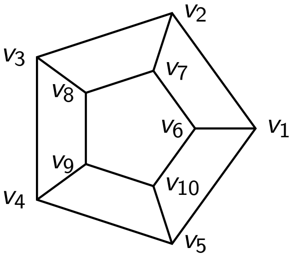
This means

$$\min_{r\text{: unit, barycentre } \mathbf{0}} \text{energy}(G, \mathbf{r}) \approx (d - \lambda_2)v(G).$$

Vertex-transitive Graphs

If for any $u, v \in V(G)$ there exists an automorphism that maps u to v, then G is called *vertex-transitive*.

An *automorphism* of a graph G is a permutation σ of $V(G)$, such that (u, v) is an edge if and only if $(\sigma(u), \sigma(v))$ is an edge.



$$\sigma = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \\ v_7 & v_8 & v_9 & v_{10} & v_6 & v_2 & v_3 & v_4 & v_5 & v_1 \end{pmatrix}$$

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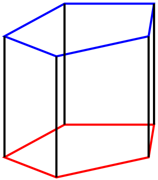

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Construction of representation:

- w is an eigenvector to eigenvalue λ_2
- w_σ is obtained from w by permuting entries w.r.t. σ
- R is a matrix with rows $(w_\sigma)_{\sigma \in Aut(G)}$

Our Semidefinite Program

Maximize: $A \bullet (V^T V)$	minimizes energy
Subject To:	
$C_v \bullet (V^T V) = 1$ for $v \in V(G)$	columns have norm 1
$J \bullet (V^T V) = 0$	origin is barycenter

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J has only 1's

Random projection of representation onto $d = 2$.

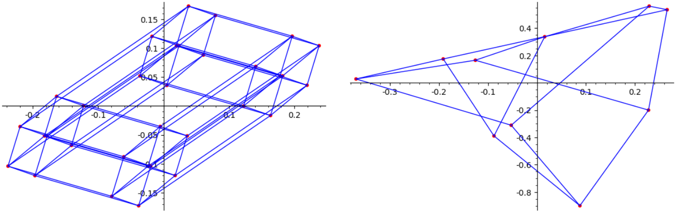

Figure 1: Hypercube for $d = 4$ and Petersen Graph

Thank you!